On Bootstrap Currents in Toroidal Systems

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Abstract

In the present paper the diffusion driven current in tokamaks and stellarators is investigated. In stellarators neoclassical bootstrap currents are serious obstacles for achieving optimum confinement since the profile of the rotational transform is modified during the rise phase of the discharge. Following the theory of axisymmetric configurations like tokamaks, predictions for nearly axisymmetric stellarators like Wendelstein VII-A yield a bootstrap current of several kilo-Amps. In Wendelstein VII-AS application of the same equations yields currents up to 70 kA, depending on the achievable β. However, this estimate does not take into account the three-dimensional geometry of the magnetic surfaces and the optimization of drift orbits in Wendelstein VII-AS. In general stellarator geometry the magnitude of the bootstrap current depends on the localization of trapped particles and the kinetic equation has to be solved in order to compute the parallel viscosity which drives the bootstrap currrent. It can be shown that in configurations with symmetry - axisymmetry, helical symmetry, quasi-helical symmetry[1]- bootstrap current and radial particle loss are proportional to each other. This relation follows directly from the momentum balance and the plasma viscosity given in Chew-Goldberger- Low formulation. This general relation, which is already known for tokamaks[2] can be derived without solving the kinetic equation. Depending on the sign of $\iota - \gamma_\omega$, the bootstrap current in helical invariant stellarators can have the opposite sign compared with tokamaks (ι is the rotational transform and γ_ω is the slope of the helical invariant lines B = const.). If the symmetry of the configuration is destroyed, it is possible to minimize the bootstrap current by a suitable choice of the magnetic field geometry.

The parallel and the perpendicular viscosities have to be calculated for the various regimes of collisionality starting from the kinetic equation. In a collision dominated plasma these effects are small. The viscous tensor of a collisional plasma given by Braginskii[3] allows one to calculate these average viscous forces for an arbitrary stellarator geometry and a geometrical factor C_b can be defined which characterizes the dependence of the bootstrap current on the topology of the magnetic surface. Such a geometrical factor also exists in the plateau regime; in the long-mean-free-path regime the bootstrap current depends on the collisionality and the radial electric field and thus a factor C_b depending on the magnetic field alone cannot be defined.

[1]Ref.[1] J. Nührenberg, R. Zille *Phys. Letters* A **129**(1988)113

[2]Ref.[2] T.A. Stringer *Plasma Phys.* **14**, (1972) 1063

[3]Ref.[3] S.I. Braginskii *Rev. of Plasma Phys. Vol. I p.250*

I. Introduction

The basic principle of stellarators is to maintain a toroidal plasma without a toroidal net current and without an external loop voltage. Since the toroidal current can be the energy reservoir for instabilities even internally generated currents like bootstrap currents should be avoided or made sufficiently small. Another reason to make bootstrap currents as small as possible in stellarators is their dependence on the plasma pressure. The bootstrap current changes the rotational transform during the heating phase of the plasma, which - especially in low shear stellarators - could generate low-order rational surfaces not existing in the vacuum configuration. Therefore the bootstrap current is an unwanted and uncontrollable component in the discharge. Stellarators without ohmic heating are well-suited to investigate the pressure driven bootstrap current, however many efforts have failed to give conclusive results and to confirm theoretical predictions. Experiments in Heliotron E[1] and Wendelstein VII-A[2] with neutral beam heating and ECRH found toroidal currents of the order of 1 - 2 kA. In these experiments it is difficult to distinguish between the bootstrap current and the current driven by the heating mechanism.

In tokamaks the bootstrap current increases the rotational transform and modifies the $q(r)$-profile and the tearing mode instability. In ohmically heated discharges measurements of the diffusion driven current are difficult, however after the application of a significant amount of auxiliary heating the existence of a bootstrap current could be shown in TFTR[3] Another reason for renewed interest in the bootstrap current is its potential of steady state operation provided the equilibrium is stable.

A large number of theoretical papers on bootstrap currents in tokamaks and stellarators exist, most of these papers are restricted to a special geometry, for example to a stellarator with one helical harmonic. Based on the review paper of Hirshman and Sigmar[4], Shaing and Callen[5] have extended the moment equation approach to non-axisymmetric configurations. As already pointed out by Wimmel[6] and Stringer[7] the neoclassical effects in the moment equations are represented by the anisotropic terms in the pressure tensor. This anisotropic pressure tensor is the macroscopic result of particle drifts, which in a plasma at low collisionality can lead to a strong deviation from a local Maxwellian. The anisotropic pressure tensor has to be calculated either from the next order moment equations or from the solution of the kinetic equation. For collisional plasmas this problem has been solved by Braginskii, who derived a general linear relationship between pressure tensor and the hydrodynamic rate of strain tensor.

In the moment equation approach an ordering scheme is employed which neglects all dissipative effects in lowest order and assumes the plasma to be characterized by a local Maxwellian without loss cone effects. In mirror machines this approximation may not be

[1]Ref.[4] S. Besshou et al. *Plasma Phys. and Contr. Fus.* **26** (1984) 565

[2]Ref.[5]

[3]Ref.[6] M.C. Zarnstoff et al. in *14th European Conf. on Contr. Fusion and Plasma Physics*, Madrid 1987 Contribut. Papers, 1, 144 (1987)

[4]Ref.[7] S.P. Hirshman, D.J. Sigmar *Nucl. Fusion* **21** (1981) 1079

[5]Ref.[8] K.C. Shaing, J.D. Callen *Phys. Fluids* **26** (1983) 3315

[6]Ref.[9] H.K. Wimmel *Nucl. Fusion* **10** (1970) 117

[7]Ref.[2]

justified, however in toroidal configurations with closed magnetic surfaces a loss cone can only occur in the perpendicular direction. Since radial drift velocities are much smaller than parallel velocities, a loss cone in the distribution function can only arise in the very long mean free path regime. Therefore, in toroidally closed configurations the anisotropy of the pressure is small and can be treated as a correction term in a perturbation technique.

In lowest order a dissipationless plasma can move freely within the magnetic surface leaving the integral poloidal flux (or the radial electric field) and the toroidal fluxes of each particle species undetermined. In first order dissipative effects are included and all quantities left undetermined in lowest order are fixed by the condition that the first order equations be integrable. These conditions are known as the flux-friction relations (Hishman, Sigmar Ref.[7]). Besides transport coefficients these flux-friction relations are the basic equations in the theory of plasma losses and bootstrap currents. They provide a linear system of algebraic equations between radial plasma losses, toroidal particle flux and radial density and temperature gradients. Elimination of the gradients allows one to correlate the toroidal fluxes linearly to the radial losses, thus yielding a linear relation between plasma losses and bootstrap current. This relation has already been discussed by Bickerton et al.[1] for tokamak plasmas; in the present paper this result will be extended to general non-axisymmetric configurations. The tokamak result can easily be generalized to all symmetric configurations like helically invariant or quasi-helical configurations. In the following, symmetric configurations are understood as configurations with *modB* containing an ignorable coordinate which implies that besides the energy another integral of particle motion exists. For these symmetric configurations the linear relationship between bootstrap current and radial plasma losses can be derived without invoking the solution of the kinetic equation, the reversal of the bootstrap current in helical stellarators, which has already been found by Shaing and Callen, is a natural and straightforward result of this theory.

The origin of the bootstrap current is the anisotropic pressure arising from the particle drifts in an inhomogeneous magnetic field thus leading to toroidal and poloidal viscous forces tangential to the magnetic surface. Heating mechanisms which generate a distortion of the distribution function in a similar fashion compete with the pressure driven toroidal currents and thus excite additional currents. On one hand this current drive mechanism is a wanted effect in order to replace the inductively driven currents in tokamaks, whereas on the other hand attention has to be paid to the influence of momentum input on radial plasma losses. In case of general toroidal geometry the flux-friction relations including external momentum sources have been studied by Coronado and Wobig[2]. In this paper this issue is discussed with special attention given to the difference between symmetric and non-symmetric configurations.

A point of particular interest is the chance to make the bootstrap current small or negligible in non-symmetric configurations. If by suitable choice of the helical field the current density changes sign across the plasma radius, the integrated bootstrap current can be made very small. In this case the rotational transform on the plasma edge is not changed by finite-β-effects, the profile of ι inside the plasma, however, varies with

[1] Ref.[10] R.J. Bickerton, J.W. Connor and J.B. Taylor *Nat. Phys. Sci.* **225** 110 (1971)

[2] Ref.[11] M. Coronado, H. Wobig *Phys. Fluids* **30** (1987) 3171

the magnitude of the bootstrap current on each magnetic surface. There is the other possibility to make the bootstrap current zero or small on each magnetic surface. If by proper choice of the Fourier spectrum of $|B|$ the *mod*-B lines are closed poloidally within one field period, trapped particles in such linked mirror configurations only drift poloidally and not toroidally. Evaluating the bootstrap current in such configurations yields a small or zero bootstrap effect on each magnetic surface. Since these configurations with localized particles resemble the tandem mirror configuration, strong radial losses of these trapped particles may occur and further investigations are needed to explore whether a magnetic field can be found where zero bootstrap current, small neoclassical losses and favourable MHD-behaviour can be combined.

The present paper discusses the relation between bootstrap current and radial plasma losses starting from the momentum balance equation of each particle species. For simplicity, first the model of an isothermal plasma with density gradients only is adopted, in this model the basic phenomena can be clearly worked out. Temperature gradient effects are included later. Many results described in the following are already described in the literature, this will not be pointed out in all cases. The aim of this paper is to give a general description of the bootstrap effect valid for any toroidal configuration. Therefore a particular coordinate system will not be used, the magnetic field and the plasma current provide a natural pair of base vectors on the magnetic surface.

II. Momentum balance equations

The best way to understand the bootstrap current in the fluid model is to start from Ohm's law which, in a plasma with single charged ions, is obtained by subtracting the two equations of momentum balance.

$$\rho_j \mathbf{v}_j \cdot \nabla \mathbf{v}_j = -\nabla p_j + e_j n_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) - \nabla \cdot \pi_j + \sum_k \mathbf{R}_{jk} \tag{2.1}$$

$j = e, i$. $\mathbf{R}_{jk} = -\mathbf{R}_{kj}$ is the momentum exchange of different particle species by Coulomb collisions. In a plasma with several particle species the interaction term is

$$\mathbf{R}_{jk} = \alpha_{jk}(\mathbf{v}_j - \mathbf{v}_k) \tag{2.2}$$

with the symmetric matrix α_{jk} being proportional to the collision frequencies ν_{jk}. In a two-fluid plasma Ohm's law is the difference of the two momentum balance equations

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \frac{1}{en}\mathbf{j} \times \mathbf{B} - \frac{\nabla p_e}{en} - \frac{\nabla \cdot \pi_e}{en} + \frac{\mathbf{R}_{ei}}{en} \tag{2.3}$$

with $n_i = n_e = n$. $\mathbf{u} = \mathbf{v}_i + m_e/m_i \mathbf{v}_e$ is the macroscopic velocity of the plasma. In this equation inertia terms have been neglected. The friction term $\mathbf{R}_{ei}$ is a linear function of the fluid velocities $\mathbf{v}_j$ and the thermal fluxes. The general form will be considered in a later chapter, here we start from the more simple formulation of a two-fluid plasma:

$$\mathbf{R}_{ei} = en\eta\,\mathbf{j} \tag{2.4}$$

η is the resistivity of the plasma which for further simplification will be taken as isotropic. Taking into account the temperature gradient would add some extra terms in Eq. (2.4), but these terms are not important to understand the basic mechanism. The effect of thermal fluxes will be investigated later.

In the Chew-Goldberger-Low[1] approximation, valid in a strong magnetic field and for all regimes of collisionality, the traceless part of the pressure tensor is

$$\pi_{ik} = (p_\parallel - p_\perp)\{b_i b_k - \frac{1}{3}\delta_{ik}\} \tag{2.5}$$

with $\mathbf{b} = \mathbf{B}/B$.

In contrast to the isotropic term of the pressure tensor, $p\,\delta_{ik}$, this anisotropic term π_{ik} leads to forces $\nabla \cdot \pi$ which are tangential to the magnetic surface, thus modifying the force balance parallel and perpendicular to the magnetic field. The neoclassical losses are calculated from the perpendicular momentum balance, whereas the bootstrap current results from the parallel momentum balance. The two terms $p_\parallel$ and $p_\perp$ have to be calculated either from the kinetic theory and the distribution function f

$$p_\parallel \propto \int v_\parallel^2 f\, d^3v \quad ; \quad p_\perp \propto \int \frac{1}{2} v_\perp^2 f\, d^3v \tag{2.6}$$

[1] Ref.[12] G.F. Chew, M.F. Goldberger, F.E. Low *Proc. R. Soc. London Ser. A* **236**, 112 (1956)

where v is the particle velocity in the frame of a displaced Maxwellian, or from the next order moment equations, which correlate $p_{||}$ and $p_{\perp}$ to the lower order moments of the distribution function. In a collisionless plasma described by the double adiabatic model $p_{||}$ and $p_{\perp}$ are functions of the magnetic surface $\psi = const$ and the magnetic field B

$$p_{||} = p_{||}(\psi, B) \quad ; \quad p_{\perp} = p_{\perp}(\psi, B) \tag{2.7}$$

<u>Plasma equilibrium</u>

Ohm's law is the basis to calculate the radial particle flux $\Gamma = \int n\mathbf{u} \cdot \mathbf{df}$ through a magnetic surface and the net toroidal current $I(\psi)$. In steady state the electric field is the gradient of a single-valued potential ϕ. To calculate $\Gamma(\psi)$ and $I(\psi)$ we assume that a plasma equilibrium with nested magnetic surfaces exists and can be described in lowest order by

$$\mathbf{J} \times \mathbf{B} = \nabla P \tag{2.8}$$

$P(\psi)$ and $N(\psi)$ are the lowest order pressure and density on the magnetic surface. This lowest order is defined by neglecting the inertial terms and the collisional terms $\nabla \cdot \pi_e$ and $\mathbf{R}_{ei}$. The lowest order plasma current is decomposed in two terms, a poloidal current $P'\mathbf{V}_o$ and a parallel current $I'(\psi)\,\mathbf{B}$

$$\mathbf{J} = P'\mathbf{V}_o + I'(\psi)\,\mathbf{B} \tag{2.9}$$

with $\nabla\cdot\mathbf{V}_o = 0$. The stream lines of the vector $\mathbf{V}_o$ are poloidally closed and the equilibrium condition requires for $\mathbf{V}_o$

$$\mathbf{V}_o \times \mathbf{B} = \nabla\psi \tag{2.10}$$

This vector $\mathbf{V}_o$ does not depend on the plasma parameters, it depends only on the geometry of the magnetic surface and can be written

$$\mathbf{V}_o = \frac{\nabla\psi \times \mathbf{B}}{B^2} + \lambda\mathbf{B} \tag{2.11}$$

with λ being calculated from $\nabla \cdot \mathbf{V}_o = 0$. $j_{||} = \lambda B$ are the Pfirsch-Schlüter currents. The vector $\mathbf{V}_o$ is correlated to the toroidal Hamada coordinate ς, since with $\mathbf{V}_o = \nabla\psi \times \nabla\nu$ the equilibrium condition demands

$$\mathbf{B} \cdot \nabla\nu = 1. \tag{2.12}$$

The lines $\nu =$ const are poloidally closed. The relation to the toroidal Hamada coordinate ς is $V'(\psi)\,\varsigma = \nu$. $V(\psi)$ is the volume inside the magnetic surface. In the following analysis we will use $\mathbf{B}$ and $\mathbf{V}_o$ as base vectors on the magnetic surface. In Ref.[8] Shaing and Callen employ the vectors $\mathbf{B}_p$ and $\mathbf{B}_t$ with $\mathbf{B} = \mathbf{B}_t + \mathbf{B}_p$, where $\mathbf{B}_t$ has toroidally closed field lines and $\mathbf{B}_p$ poloidally closed lines. The correlation to $\mathbf{V}_o$ is

$$\mathbf{B}_p = -\frac{1}{V'(\psi)}\iota\mathbf{V}_o. \tag{2.13}$$

Locally the vectors $\mathbf{V}_o$ and $\mathbf{B}$ are not perpendicular to each other, the surface averaged scalar product $< \mathbf{V}_o \cdot \mathbf{B} >$ is equal to the toroidal current $I'(\psi)$ inside a magnetic surface, as will be shown in the next chapter. Therefore the choice of $\mathbf{V}_o$ and $\mathbf{B}$ as base vectors on the magnetic surface is particularly convenient when equations for the toroidal current are established. Furthermore, in most cases terms with $< \mathbf{V}_o \cdot \mathbf{B} >$ are negligible in the resulting equations making these shorter and more comprehensible.[1]

The toroidal current between two adjacent magnetic surfaces is

$$dI_{tor} = P' \int_{F_1} \mathbf{V}_o \cdot \mathbf{df} + I' \int \mathbf{B} \cdot \mathbf{df} = I'(\psi)\, d\psi \tag{2.14}$$

The first integral is zero since $\mathbf{V}_o$ carries no toroidal flux and the toroidal current between two adjacent magnetic surfaces is given by $I'(\psi)\, d\psi$.

The poloidal current dI_{pol} is calculated from

$$\begin{aligned} dI_{pol} &= P' \int_{F_2} \mathbf{V}_o \cdot \mathbf{df} + I' \int_{F_2} \mathbf{B} \cdot \mathbf{df} \\ &= -P' V'(\psi)\, d\psi + I' \iota\, d\psi \end{aligned} \tag{2.15}$$

$d\psi$ is the toroidal magnetic flux and $\iota d\psi$ the poloidal magnetic flux.

The lowest order particle flow of each species is defined by

$$0 = -\nabla P_j + e_j N_j \left(\mathbf{E} + \mathbf{V}_j \times \mathbf{B} \right) \tag{2.16}$$

which under steady state conditions yields

$$\mathbf{V}_j = E_j(\psi)\, \mathbf{V}_o + \Lambda_j(\psi)\, \mathbf{B} \tag{2.17}$$

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[1] In Hamada coordinates V, η, ς the magnetic field vector is

$$\mathbf{B} = \psi'(V)\, \nabla V \times \nabla \eta - \chi'(V)\, \nabla V \times \nabla \varsigma.$$

η is the poloidal coordinate and ς the toroidal Hamada coordinate. The Jacobian is unity and $\mathbf{B} \cdot \nabla \varsigma = 1/V'(\psi)$; $\mathbf{B} \cdot \nabla \eta = \iota / V'(\psi)$. The following relation

$$\mathbf{B}_p \times \mathbf{B} = -\frac{\iota}{V'(\psi)} \nabla \psi$$

leads to Eq. (2.13). In Hamada coordinates the operators $\mathbf{B} \cdot \nabla$ and $\mathbf{V}_o \cdot \nabla$ are written:

$$\mathbf{B} \cdot \nabla = \frac{1}{V'(\psi)} \left(\iota \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \varsigma} \right) \quad ; \quad \mathbf{V}_o \cdot \nabla = -\frac{\partial}{\partial \eta} \tag{A2.1}$$

The volume element is $V'(\psi) d\eta d\varsigma d\psi$ and the surface element

$$\frac{df}{|\nabla \psi|} = V'(\psi) d\eta d\varsigma \tag{A2.2}$$

These relations will be used in the further analysis.

with

$$E_j(\psi) = \Phi'(\psi) + \frac{kT_j}{e_j}\frac{P_j'(\psi)}{P_j} \tag{2.18}$$

$\Phi'(\psi)$ is the ψ-derivative of the electric potential. In analogy to the electric current the flux of $\mathbf{V}_j$ can be separated in a toroidal and poloidal component

$$\begin{aligned} u_t\, d\psi &= d\int_{F_1} \mathbf{V}_j \cdot \mathbf{df} = \Lambda_j(\psi)\, d\psi \\ u_p\, d\psi &= d\int_{F_2} \mathbf{V}_j \cdot \mathbf{df} = -E_j(\psi)V'(\psi)\, d\psi + \Lambda_j(\psi)\, \iota\, d\psi \end{aligned} \tag{2.19}$$

The poloidal flux u_p consists of the diamagnetic flux, a flux of the $\mathbf{E}\times\mathbf{B}$-drift motion and part of the toroidal flux which flows poloidally by following the magnetic field lines. The toroidal flux is determined by $\Lambda_j(\psi)$ only. In a plasma with several species of ions the toroidal flux of each species has to be calculated in order to find the resulting toroidal current $I'(\psi)$

$$I'(\psi) = \sum e_j N_j \Lambda_j(\psi) \tag{2.20}$$

which in a two-fluid system is

$$I'(\psi) = -eN\left(\Lambda_e(\psi) - \Lambda_i(\psi)\right) \tag{2.21}$$

In lowest order the functions P_j, T_j and Φ are functions of the magnetic surface $\psi =$ const. only. The first order corrections p_1, T_1 and Φ_1 have to be calculated from the first order equations. These equations are magnetic differential equations for these quantities and the first order plasma flow $\mathbf{v}_1$, which describes the radial loss through a magnetic surface, can be found after p_1, Φ_1 are known. The integral fluxes, however, can be found from integrating the first order equations over the magnetic surface.

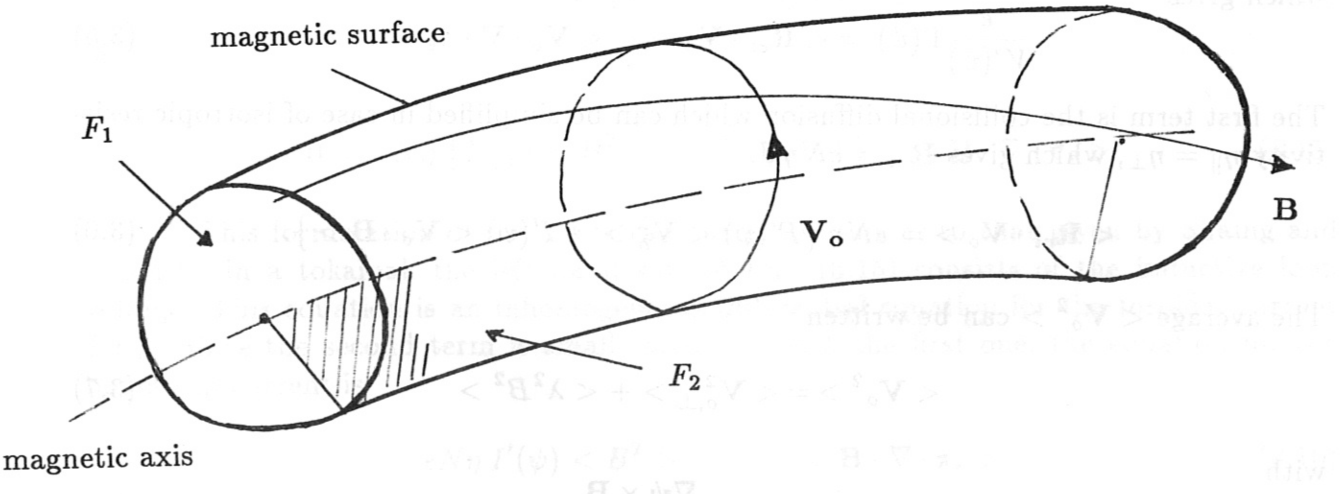

Fig. 1: Magnetic surface of a toroidal equilibrium with the two base vectors $\mathbf{B}$ and $\mathbf{V}_o$.

III. The flux-friction relations

A correlation between the particle flux Γ, the toroidal current $I(\psi)$ and the averaged viscous forces is provided by the flux- friction relations[1]. To get these relations we multiply Ohm's law with $\mathbf{B}$ or $\mathbf{V}_o$ and average over the magnetic surface. Averaging over the magnetic surface is defined by

$$< g >= \frac{\int g \frac{df}{|\nabla\psi|}}{\int \frac{df}{|\nabla\psi|}} \quad ; \quad V'(\psi) =: \int \frac{df}{|\nabla\psi|} \tag{3.1}$$

Since in a stellarator the electric potential is single-valued we find

$$< \mathbf{B} \cdot \nabla\phi >= 0 \qquad < \mathbf{V}_o \cdot \nabla\phi >= 0 \tag{3.2}$$

and

$$< \mathbf{B} \cdot \frac{\nabla p_e}{n} >= 0 \qquad < \mathbf{V}_o \cdot \frac{\nabla p_e}{n} >= 0 \tag{3.3}$$

in lowest and first order. Only second order terms $< \mathbf{B} \cdot n_1 \nabla p_{e,1} >$ could give a finite contribution. n_1 and $p_{e,1}$ are the variations of density and pressure in the magnetic surface. Because of the conservation of charge we have $< \mathbf{V}_o \cdot (\mathbf{j} \times \mathbf{B}) >= 0$ in all orders. The particle flux through a magnetic surface is

$$\Gamma = \int N\mathbf{u} \cdot \mathbf{df} = N < \mathbf{V}_o \cdot (\mathbf{u} \times \mathbf{B}) > V'(\psi) \tag{3.4}$$

With these results we multiply Ohm's law with $\mathbf{V}_o$ and average over the magnetic surface which gives

$$\frac{e}{V'(\psi)}\Gamma(\psi) = < \mathbf{R}_{ei} \cdot \mathbf{V}_o > - < \mathbf{V}_o \cdot \nabla \cdot \pi_e > . \tag{3.5}$$

The first term is the collisional diffusion which can be simplified in case of isotropic resistivity $\eta_{\parallel} = \eta_{\perp}$, which gives $\mathbf{R}_{ei} = eN\eta\,\mathbf{J}$.

$$< \mathbf{R}_{ei} \cdot \mathbf{V}_o >= eN\eta \left\{ P'(\psi) < \mathbf{V}_o^2 > + I'(\psi) < \mathbf{V}_o \cdot \mathbf{B} > \right\}. \tag{3.6}$$

The average $< {\mathbf{V}_o}^2 >$ can be written

$$< {\mathbf{V}_o}^2 >=< \mathbf{V}_{o,\perp}^2 > + < \lambda^2 B^2 > \tag{3.7}$$

with

$$\mathbf{V}_{o,\perp} = \frac{\nabla\psi \times \mathbf{B}}{B^2}. \tag{3.8}$$

[1] see: Ref.[7] S.C. Hirshman, D.J. Sigmar *Nucl. Fusion* **21**, 1079 (1981)

We define the Pfirsch-Schlüter factor D_o by

$$D_o = \frac{< \mathbf{V}_o^2 >}{< \mathbf{V}_{o,\perp}^2 >} = 1 + \frac{< \lambda^2 B^2 >}{< \mathbf{V}_{o,\perp}^2 >} \tag{3.9}$$

which is unity in case of zero Pfirsch-Schlüter currents. This is only possible if $\mathbf{V}_o \cdot \nabla B = 0$ which is the characteristic feature of isodynamic equilibria[2] . The following rough estimate holds

$$< \mathbf{V}_{o,\perp}^2 > \quad = < \frac{|\nabla \psi|^2}{B^2} > \approx r^2 \tag{3.10}$$

where r is the average radius of the magnetic surface. The factor $< \mathbf{V}_o \cdot \mathbf{B} >$ is equal to the toroidal current $I(\psi)$, which can be seen from the definition of j yielding

$$< \mathbf{B} \cdot \mathbf{j} > \quad = P'(\psi) < \mathbf{V}_o \cdot \mathbf{B} > + I'(\psi) < B^2 > \tag{3.11}$$

and the expression for $< \mathbf{B} \cdot \mathbf{j} >$ derived by Kruskal and Kulsrud[1]

$$< \mathbf{B} \cdot \mathbf{j} > = P'(\psi)\, I(\psi) + I'(\psi) < B^2 > . \tag{3.12}$$

Thus, the net particle flow through a magnetic surface is

$$\frac{e\Gamma}{V'(\psi)} = eN\eta \{ P'(\psi) D_o < \frac{|\nabla\psi|^2}{B^2} > + I' I \} - < \mathbf{V}_o \cdot \nabla \cdot \pi_e > . \tag{3.13}$$

The first term on the right hand side is the Pfirsch-Schlüter diffusion, the second one is the classical Pinch effect and the third term contains the neoclassical diffusion.

Application of the same averaging method to $\mathbf{B}\cdot$ eq. (2.3) gives the parallel momentum balance.

$$0 = < \mathbf{R}_{ei} \cdot \mathbf{B} > - < \mathbf{B} \cdot \nabla \cdot \pi_e > \tag{3.14}$$

or

$$0 = eN\eta \{ I'(\psi) < B^2 > + P'(\psi)\, I(\psi) \} - < \mathbf{B} \cdot \nabla \cdot \pi_e > \tag{3.15}$$

This formulation of the bootstrap current is similar to that given by Shaing and Callen[2]. In a tokamak the left hand side of Eq. (3.15) consists of the inductive loop voltage. This equation is an inhomogeneous differential equation for the toroidal current $I(\psi)$. Since the second term is small compared with the first one, the equation for the bootstrap current is

$$eN\eta\, I'(\psi) < B^2 > \quad = \quad < \mathbf{B} \cdot \nabla \cdot \pi_e > \tag{3.16}$$

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[2] Ref.[13] D. Palumbo, *Nuovo Cimento* XB 53 (1968) 507

[1] Ref.[14] M.D. Kruskal, R.M. Kulsrud, *Phys. Fluids* **1**, (1958) 265

[2] Ref.[8]

Without neoclassical (or viscous) effects the bootstrap current in stellarators would be zero and the particle loss is given by the Pfirsch-Schlüter diffusion. In the neoclassical approximation the toroidal current is finite and adds an extra term to the particle flux. Since $I^2(\psi)$ is a function increasing with ψ, this current driven diffusion flux is directed inwards and opposite to the Pfirsch-Schlüter flux.

These equations seem to be asymmetric in the particle species, however this is not the case. From the symmetry of the Coulomb collisions and the condition of ambipolarity it follows that

$$\begin{aligned} < \mathbf{V}_o \cdot \nabla \cdot \pi_e > + < \mathbf{V}_o \cdot \nabla \cdot \pi_i > &= 0 \\ < \mathbf{B} \cdot \nabla \cdot \pi_e > + < \mathbf{B} \cdot \nabla \cdot \pi_i > &= 0 \,. \end{aligned} \tag{3.17}$$

These equations say that in stellarators the sum of the averaged tangential viscous forces in the poloidal and toroidal directions is zero. Using the Chew-Goldberger-Low form of the pressure tensor, these averaged viscous forces can be written

$$\begin{aligned} < \mathbf{V}_o \cdot \nabla \cdot \pi_e > &= - < (p_{\|} - p_{\perp})_e \mathbf{V}_o \cdot \frac{\nabla B}{B} > \\ < \mathbf{B} \cdot \nabla \cdot \pi_e > &= - < (p_{\|} - p_{\perp})_e \mathbf{B} \cdot \frac{\nabla B}{B} > \,. \end{aligned} \tag{3.18}$$

This formulation of the neoclassical effects shows that particle flux and bootstrap current are correlated to the inhomogeneity of the magnetic field on the magnetic surface, the particle flux to the poloidal variation of B and the bootstrap current to the parallel variation of B. Since B is a function of two variables - poloidal and toroidal angle - these two forces can behave differently depending on the particular configuration. In tokamaks the magnetic field depends only on the poloidal angle and both forces are proportional to each other.

Neoclassical diffusion is zero if $\mathbf{V}_o \cdot \frac{\nabla B}{B} = 0$. It can be shown that this condition implies zero Pfirsch- Schlüter currents. Therefore it is expected that in configurations with reduced Pfirsch-Schlüter currents all neoclassical effects including the bootstrap current are small. Radial losses in stellarators are particularly large, if the trapped particles experience a radial drift leading to non-confined orbits. These losses are described by the bounce-averaged distribution function $\overline{f}$. As Shaing and Callen[1] have shown this bounce-averaged distribution function does not contribute to the driving term $< (p_{\|} - p_{\perp}) \frac{\mathbf{B} \cdot \nabla B}{B} >$ of the bootstrap current. Therefore, in stellarators the bootstrap current can be small although particle losses are large.

The main problem in computing the bootstrap current is to determine the anisotropy of the pressure. This can be either done by solving the higher order moment equations or by solving the kinetic equation for the distribution function. In a collisionless plasma described by the double adiabatic theory, a solution of the equilibrium condition is, when $p_{\|}$ and $p_{\perp}$ are functions of the magnetic surface and the magnetic field B,

$$p_{\|} = p_{\|}(\psi, B) \quad ; \quad p_{\perp} = p_{\perp}(\psi, B). \tag{3.19}$$

[1] Ref.[8]

Under these conditions there are no neoclassical effects since for any function $g(\psi, B)$

$$\begin{aligned} < g(\psi, B)\, \mathbf{V}_o \cdot \nabla B > &= 0 \\ < g(\psi, B)\, \mathbf{B} \cdot \nabla B > &= 0 \end{aligned} \tag{3.20}$$

Collisional effects must be included in order to obtain nonvanishing effects.

The parallel viscous force is equivalent to an electric field, which drives the bootstrap current. This surface averaged electric field is defined

$$E_{eff} = \frac{1}{eNB_o} < \mathbf{B} \cdot \nabla \pi_e > \tag{3.21}$$

and B_o is a reference field. The equivalent loop voltage is

$$U_{eff} = 2\pi R\, E_{eff}. \tag{3.22}$$

A rough estimate of this loop voltage can be found if we approximate

$$\frac{\mathbf{B} \cdot \nabla B}{B} \approx \frac{B_o}{R} \tag{3.23}$$

which gives

$$U_{eff} \approx 2\pi \frac{kT}{e} \left(\frac{p_{\|} - p_{\perp}}{P}\right)_{max} g \tag{3.24}$$

where g is a factor of order unity or less. This estimate shows that in a plasma with a temperature of several keV a very small anisotropy of the pressure $(p_{\|} - p_{\perp})/P$ of the order 10^{-3} is sufficient to generate a loop voltage of several volts.

IV. Bootstrap current in symmetric systems.

As shown above, the bootstrap currents and the neoclassical fluxes are integrals over the difference of parallel and perpendicular pressure. This raises the question, whether there are closer connections between these two quantities. In general non-axisymmetric configurations this relation cannot be found without calculating the anisotropic pressure explicitly. In configurations with symmetry, however, such a relation exists. Axisymmetric tokamaks and helically invariant stellarators are examples of configurations with an ignorable coordinate. If u, v are toroidal and poloidal coordinates, a linear combination $\omega = \gamma_1 u + \gamma_2 v$ exists, with $B = B(\omega)$. The magnetic field lines in this coordinate system are described by

$$\chi = C_1 u + C_2 v + \chi_p(\omega) \tag{4.1}$$

and the stream lines of $\mathbf{V}_o$ by

$$\varsigma = C_3 u + \varsigma_p(\omega)\,. \tag{4.2}$$

χ_p and ς_p are periodic functions of ω. In magnetic coordinates u, v the field lines are straight lines and $\chi_p = 0$.

Now, it can be shown that in these symmetric systems the two terms $\mathbf{V}_o \cdot \nabla B$ and $\mathbf{B} \cdot \nabla B$ are proportional to each other with the factor being a constant on the magnetic surface. In order to show this, we introduce two new base vectors $\mathbf{U}$ and $\mathbf{V}$ by rotating the the old base vectors

$$\begin{aligned} \mathbf{U} &= a_{11}\,\mathbf{V}_o + a_{12}\,\mathbf{B} \\ \mathbf{V} &= a_{21}\,\mathbf{V}_o + a_{22}\,\mathbf{B} \end{aligned} \tag{4.3}$$

where the determinant of the matrix a_{ik} is 1. The vector $\mathbf{U}$ is chosen to be parallel to the lines $\omega = const$

$$\mathbf{U} = \nabla\psi \times \nabla\omega \quad \text{and} \quad \mathbf{U} \cdot \nabla\omega = 0\,. \tag{4.4}$$

With $\mathbf{B} = \nabla\psi \times \nabla\chi$ and $\mathbf{V}_o = \nabla\psi \times \nabla\varsigma$ this leads to the condition

$$0 = \mathbf{U} \cdot \nabla B = \frac{dB}{d\omega} \cdot \{\nabla\psi \cdot (\nabla\omega \times \nabla(a_{11}\varsigma + a_{12}\chi))\}\,. \tag{4.5}$$

The term $a_{11}\varsigma + a_{12}\chi$ is a function of ω if we make the choice

$$a_{11}C_3 + a_{12}C_1 = \gamma_1 \quad , \quad a_{12}C_2 = \gamma_2 \tag{4.6}$$

$\iota = C_1/C_2$ is the rotational transform of the field lines and $\gamma_\omega = \gamma_1/\gamma_2$ the slope of the invariant direction $\omega = const$. The solution is

$$\begin{aligned} a_{12} &= a_{21} = \frac{\gamma_2}{C_2} \\ a_{11}C_3 &= \gamma_1 - \gamma_2\iota = \gamma_2(\gamma_\omega - \iota)\,. \end{aligned} \tag{4.7}$$

a_{22} is determined by $|a_{ik}| = 1$. The matrix a_{ik} is constant on the magnetic surface and from Eq. (4.3) the following result is derived

$$a_{11}\,\mathbf{V}_o \cdot \nabla B = -a_{12}\,\mathbf{B} \cdot \nabla B$$

or

$$\mathbf{B} \cdot \nabla B = \frac{C_2}{C_3}(\iota - \gamma_\omega)\mathbf{V}_o \cdot \nabla B\,. \tag{4.8}$$

In deriving this relation we used the invariance of the magnetic field lines and the current lines, which implies that the magnetic surface is invariant in direction of $\mathbf{U}$. In quasi-helical configurations[1] the magnetic field B depends on the coordinate ω, when it is written in magnetic coordinates u, v. The magnetic surfaces need not be helically invariant and therefore this kind of configuration can also exist in toroidal geometry. In magnetic coordinates $\chi = C_1\,u + C_2\,v$ and $\varsigma = C_3\,u + \chi_p(\omega)$. This form of the stream function ς follows from

$$\mathbf{B} \cdot \nabla\varsigma = 1 \longrightarrow \{\frac{d}{du} + \iota\frac{d}{dv}\}\,\varsigma = \frac{1}{B^2(\omega)} \tag{4.9}$$

[1] J. Nührenberg, R. Zille , Ref.[1]

which means that in quasi-helical configurations the current lines are helically invariant in magnetic coordinates.

An immediate consequence of the relation (4.8) is the proportionality of bootstrap current and poloidal viscous force

$$< (p_{\|} - p_{\perp})_e \mathbf{B} \cdot \frac{\nabla B}{B} > = \frac{C_2}{C_3}(\iota - \gamma_\omega) < (p_{\|} - p_{\perp})_e \frac{\mathbf{V}_o \cdot \nabla B}{B} > . \qquad (4.10)$$

The same relation holds for the ions. Using this result the bootstrap current can be expressed in terms of the neoclassical particle flux Γ_{neo} which is the difference of the particle flux Γ minus the Pfirsch-Schlüter term $< \mathbf{R}_{ei} \cdot \mathbf{V}_o >$.

$$\begin{aligned} N\eta I' < B^2 > &= -\frac{C_2}{C_3}(\iota - \gamma_\omega) < (p_{\|} - p_{\perp}) \frac{\mathbf{V}_o \cdot \nabla B}{B} > \\ N\eta I' < B^2 > &= -\frac{C_2}{C_3}(\iota - \gamma_\omega) \frac{1}{V'} \Gamma_{neo}(\psi) . \end{aligned} \qquad (4.11)$$

This equation shows that from tokamaks to helically invariant stellarators the bootstrap current changes its sign. In tokamaks the invariant direction is toroidal and $\gamma_\omega = 0$. In helically invariant stellarators (Heliac and quasi-helical configurations) the slope of the field lines in general is smaller than the slope of the helically invariant direction, and therefore $\iota < \gamma_\omega$. The bootstrap current flows in the opposite direction. If the slope in much larger than the rotational tranform, the magnitude of the bootstrap current is much larger than in the equivalent tokamak with the same neoclassical particle loss.

This reversal of the bootstrap current can be understood on the basis of drift orbits of trapped particles. In tokamaks the drift of the banana orbits is toroidal, in helically invariant or quasi-helical stellarators these trapped particles drift poloidally and toroidally along the lines $\omega = const$. However, the toroidal drift changes its sign with $\iota - \gamma_\omega$ and the trapped particles drift in opposite directions than those in a tokamak. Since the bootstrap current is the result of collisional interaction of trapped particles with circulating particles, its sign depends on the toroidal drift of trapped particles.

The parallel viscous force driving the bootstrap current can be interpreted as an electric field, this equivalent electric field is defined by Eq. (3.21). The corresponding loop voltage in symmetric configurations is

$$\begin{aligned} U_{eff} &= \frac{2\pi R}{eNB_o} < (p_{\|} - p_{\perp})_e \frac{\mathbf{B} \cdot \nabla B}{B} > \\ &= \frac{2\pi R}{NB_o} \frac{C_2}{C_3}(\iota - \gamma_\omega) \frac{1}{V'(\psi)} \Gamma_{neo} . \end{aligned} \qquad (4.12)$$

The coefficient C_2 is equal to the ψ-derivative of the toroidal magnetic flux

$$C_2(\psi)\, d\psi = d \int_{\psi = const} \mathbf{B} \cdot d\mathbf{f} = d\Phi(\psi) \qquad (4.13)$$

and

$$C_3(\psi) = \int_{v=const} \frac{dl}{B} = \frac{2\pi R}{B_1} . \tag{4.14}$$

This equation defines an average field B_1. The effective loop voltage can be written

$$U_{eff} = \frac{B_1}{B_o}(\iota - \gamma_\omega)\frac{d\Phi}{dV}\frac{\Gamma(\psi)}{N(\psi)} \tag{4.15}$$

The magnetic field B_1 can be set equal to B_o. Defining the magnetic field $\bar{B}$ by

$$\frac{d\Phi}{dV} = \frac{\bar{B}}{2\pi R}$$

gives for the effective loop voltage

$$U_{eff} = \frac{\bar{B}}{2\pi R}(\iota - \gamma_\omega)\frac{\Gamma_{neo}}{N(\psi)} . \tag{4.16}$$

The particle flux can be replaced by defining a neoclassical confinement time τ_p by

$$\Gamma_{neo}(\psi) =: \frac{\int N\, dv}{\tau_p} = \frac{\bar{N} 2\pi^2 R r^2}{\tau_p}$$

$\bar{N}$ is the averaged density inside the magnetic surface $\psi = const$ and r is the averaged radius of this magnetic surface defined by its volume V. With these definitions we can write the effective loop voltage

$$U_{eff} = \bar{B}\pi r^2(\iota - \gamma_\omega)\frac{\bar{N}}{\tau_p N} . \tag{4.17}$$

$\bar{N}/N$ is a profile factor of the order 1 - 2.

As an example we consider Wendelstein VII-A, where the helical fields are small making the axisymmetric approximation applicable. The data are: $\iota = 0.5, \bar{B} = 3\,\mathrm{T}, r = 0.08\,\mathrm{m}$. The effective loop voltage is

$$U_{eff} = 3 \cdot 10^{-2}\,\frac{\bar{N}}{N}\frac{1}{\tau_p} \quad [\mathrm{Volt}] . \tag{4.18}$$

With $\bar{N}/N \approx 2$ and a neoclassical confinement time of $\tau_p = 0.1$ - 0.2 sec, we obtain

$$U_{eff} \approx 0.3 - 0.6 \quad [\mathrm{Volt}] . \tag{4.19}$$

This a local loop voltage, it only exists on magnetic surfaces with a finite particle flux Γ. Since on the magnetic axis the particle flux is zero, the bootstrap current is zero on the magnetic axis. If the particle deposition profile is centered to the magnetic axis, the particle flux $\Gamma(\psi) = \int S\, dV$ is nearly constant over the plasma radius and the effective loop voltage drives a large bootstrap current. If particle refuelling mainly takes place in the boundary region, which happens in a high density plasma with cold gas refuelling, the particle source term is zero in the bulk of the plasma and the flux vanishes.

In a neoclassical tokamak with $B = 2\mathrm{T}$, $a = 0.4$ m, $0.5 < \iota < 1.0$ and a neoclassical confinement time of 0.5 - 1.0 sec the effective loop voltage is

$$U_{eff} \approx 1.3 - 2.6 \quad [\mathrm{Volt}] . \tag{4.20}$$

V. Comparison with tokamak

The equations derived in the preceding section are valid for all symmetric configurations. We shall compare these results with the well-known neoclassical equations of tokamaks. For this purpose we begin with the neoclassical expressions for particle losses and the bootstrap current in the banana regime.

$$N\bar{v}_D = -D\,N' = \nu\frac{{\rho_e}^2}{\iota^2}(\frac{R}{r})^{3/2}\,N' \tag{5.1}$$

is the average particle flux through a magnetic surface and

$$j_b = (\frac{R}{r})^{1/2}\frac{kT}{\iota B_o}\,N' \tag{5.2}$$

is the average current density of the bootstrap current. As already shown by Bickerton[1] et al. and Stringer[2] the correlation between particle flux and bootstrap current can be formulated as follows

$$N\eta j_b = \frac{r}{R}B_o \iota N\bar{v}_D \tag{5.3}$$

In these equations r is the average radius of the magnetic surface under consideration. In order to compare this result with equation (4.11) we use the following approximations

$$\frac{C_2}{V'} \approx \frac{B_o}{2\pi R} \quad , \quad C_3 \approx \frac{2\pi R}{B_o} \tag{5.4}$$

$$I'\,B_o \approx j_b \quad , \quad \Gamma_{neo} \approx (2\pi)^2 rR\,N\bar{v}_D .$$

With these approximations Eq. (4.11) gives

$$N\eta\; j_b = C\,B_o\frac{r}{R}\,(\iota - \gamma_\omega)\,N\bar{v}_D\;. \tag{5.5}$$

C is a constant of the order one which summarizes the various geometrical factors in C_3, C_2, I' and Γ_{neo}. This equation coincides with the tokamak result if we omit the term γ_ω. Pytte and Boozer[3] have derived a similar equation for the helically symmetric stellarator. Instead of the factor $\iota - \gamma_\omega$ these authors only find the factor γ_ω. Since ι is small compared with γ_ω, the difference is not significant in magnitude, however, the difference in direction of the bootstrap current is not seen in this approximation.

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[1] Ref.[10] R.J. Bickerton, J.W. Connor and J.B. Taylor *Nat. Phys. Sci.* **225** 110 (1971)

[2] Ref.[2] Plasma Phys. **14**, 1063 (1972)

[3] Ref.[15] A. Pytte, A.H. Boozer *Phys. Fluids* **24**(1981) 88

VI. Non-symmetric configurations

In configurations with symmetry the proportionality between poloidal and parallel viscous forces is the reason for the coupling between the radial losses and the bootstrap current. In configurations without symmetry this is no longer the case. Locally, the term $\mathbf{V}_o \cdot \nabla B$ and $\mathbf{B} \cdot \nabla B$ are proportional, however the factor depends on the position on the magnetic surface and relation (4.8) is not valid anymore. In general it is difficult to find the relation between parallel and poloidal viscous forces since the solution of the kinetic equation is needed in order to compute the moments $p_{\|}$ and $p_{\perp}$. As shown in Eqs. (3.13) and (3.18) the neoclassical particle flux is proportional to $< (p_{\|} - p_{\perp})_e \mathbf{V}_o \cdot \frac{\nabla B}{B} >$ and the bootstrap current proportional to $< (p_{\|} - p_{\perp})_e \mathbf{B} \cdot \frac{\nabla B}{B} >$. In a Hamada coordinate system η, ς these terms are

$$
\begin{aligned}
< (p_{\|} - p_{\perp})_e \mathbf{V}_o \cdot \frac{\nabla B}{B} > \quad &= \quad - < (p_{\|} - p_{\perp})_e \frac{\partial}{\partial \eta} ln\, B > \\
< (p_{\|} - p_{\perp})_e \mathbf{B} \cdot \frac{\nabla B}{B} > \quad &= \quad \frac{1}{V'(\psi)} \; < (p_{\|} - p_{\perp})_e (\iota \frac{\partial}{\partial \eta} + \frac{\partial}{\partial \varsigma}) ln\, B >
\end{aligned}
\tag{6.1}
$$

In an axisymmetric tokamak the toroidal derivative $\frac{\partial}{\partial \varsigma}$ is zero which leads to Eq. (4.11). In stellarators without symmetry $ln\, B$ depends on both coordinates and the Fourier expansion of $ln\, B$ is

$$
ln\, B = \sum_{l,m} a_{l,m} \, cos\, (l\eta - m\varsigma) \tag{6.2}
$$

$$
\begin{aligned}
< (p_{\|} - p_{\perp})_e \mathbf{V}_o \cdot \frac{\nabla B}{B} > &= - \sum_{l,m} a_{l,m}\, l < (p_{\|} - p_{\perp})_e \, sin\, (l\eta - m\varsigma) \\
< (p_{\|} - p_{\perp})_e \mathbf{B} \cdot \frac{\nabla B}{B} > &= - \sum_{l,m} a_{l,m} \, (l\iota - m) \; < (p_{\|} - p_{\perp})_e \, sin\, (l\eta - m\varsigma) >
\end{aligned}
\tag{6.3}
$$

If the magnetic field only consists of one helical harmonic $l = L$ and $m = M$, this immediately yields Eq. (4.11) again. In order to cancel the bootstrap current to zero, besides the leading harmonic L, M several other harmonics must exist so that

$$
\iota < (p_{\|} - p_{\perp})_e \frac{\partial}{\partial \eta} ln\, B > \quad = \quad - < (p_{\|} - p_{\perp}) \frac{\partial}{\partial \varsigma} ln\, B > \tag{6.4}
$$

Collision dominated plasma

To analyse the surface averaged quantities further, we consider a collision dominated plasma first, where these terms can be calculated explicitly. In a collision dominated plasma the viscous forces are known from the Braginskii theory[1]. The viscous tensor consists of 5 terms describing bulk viscosity, shear-viscosity and gyro-viscosity. The leading

[1] Ref.[3]

term is the bulk viscosity which yields the following equation for the difference $p_{\|} - p_{\perp}$ for each particle species

$$p_{\|} - p_{\perp} \; = \; -3\tau P\big\{b_i b_k \frac{\partial V_i}{\partial x_k} - \frac{1}{3}\frac{\partial V_k}{\partial x_k}\big\} \tag{6.5}$$

(summation over equal indices). τ is the self collision time. The lowest order flow velocity of the plasma is the diamagnetic drift and the $\mathbf{E} \times \mathbf{B}$-drift within the magnetic surface. Furthermore, in lowest order an undetermined toroidal flux $\Lambda(\psi)\,\mathbf{B}$ exists. Thus, the lowest order flow velocity from equation (2.15) (without inertial terms, $< \mathbf{V}_o \cdot \nabla \cdot \pi_e >$ - terms and $\mathbf{R}_{ei}$) is

$$\mathbf{V}_j \; = \; E_j(\psi)\mathbf{V}_o + \Lambda_j \mathbf{B} \tag{6.6}$$

with

$$E_j(\psi) \; = \; \frac{kT}{e_j}\frac{N_j'}{N_j} + \Phi'(\psi). \tag{6.7}$$

With the aid of these equations the averaged viscous forces can be evaluated

$$\begin{aligned}
< \mathbf{V}_o \cdot \nabla \cdot \pi_e > &= - < (p_{\|} - p_{\perp})_e (\mathbf{V}_o \cdot \frac{\nabla B}{B}) > \\
& 3\tau_e P\big\{E_e(\psi) < (\mathbf{V}_o \cdot \frac{\nabla B}{B})^2 > + \Lambda_e < (\mathbf{V}_o \cdot \frac{\nabla B}{B})(\mathbf{B} \cdot \frac{\nabla B}{B}) > \big\} \\
< \mathbf{B} \cdot \nabla \cdot \pi_e > &= - < (p_{\|} - p_{\perp})_e (\mathbf{B} \cdot \frac{\nabla B}{B}) > \\
&= 3\tau_e P\big\{E_e(\psi) < (\mathbf{V}_o \cdot \frac{\nabla B}{B})(\mathbf{B} \cdot \frac{\nabla B}{B}) > + \Lambda_e < (\mathbf{B} \cdot \frac{\nabla B}{B})^2 > \big\}.
\end{aligned} \tag{6.8}$$

For ions corresponding equations exist. After defining the coefficients

$$\begin{aligned}
C_p &= < (\mathbf{V}_o \cdot \frac{\nabla B}{B})^2 > \quad ; \quad C_t = < (\mathbf{B} \cdot \frac{\nabla B}{B})^2 > \\
C_b &= < (\mathbf{B} \cdot \frac{\nabla B}{B})(\mathbf{V}_o \cdot \frac{\nabla B}{B}) >
\end{aligned} \tag{6.9}$$

the ambipolar condition and the parallel momentum balance Eq. (3.17) can be written

$$\begin{aligned}
C_p \left\{\tau_e E_e + \tau_i E_i\right\} + C_b \left\{\tau_e \Lambda_e + \tau_i \Lambda_i\right\} &= 0 \\
C_b \left\{\tau_e E_e + \tau_i E_i\right\} + C_t \left\{\tau_e \Lambda_e + \tau_i \Lambda_i\right\} &= 0
\end{aligned} \tag{6.10}$$

This is a homogeneous system for $X =: \tau_e E_e + \tau_i E_i$ and $Y =: \tau_e \Lambda_e + \tau_i \Lambda_i$ which in case of $C_p C_t - C_b^2 \neq 0$ has the trivial solution $X = 0$ and $Y = 0$ or

$$\Lambda_i = -\frac{\tau_e}{\tau_i}\Lambda_e \quad ; \quad E_i = -\frac{\tau_e}{\tau_i}E_i$$

In a two-fluid plasma the toroidal current is

$$I'(\psi) \; = \; -eN\left(\Lambda_e(\psi) - \Lambda_i(\psi)\right) \tag{6.11}$$

and from Eqs. (3.16),(3.17) and (6.8) the toroidal fluxes $\Lambda_j(\psi)$ are calculated in terms of E_e and E_i and finally the toroidal current is given by

$$I'(\psi) = C_b 3P \frac{\tau_e \tau_i (E_e - E_i)}{< B^2 > eN\eta(\tau_e + \tau_i) + 3\frac{kT}{e}\tau_e\tau_i C_t} \tag{6.12}$$

The geometrical factor C_b is the relevant coefficient for the bootstrap current; if this term is zero the parallel viscosity $< \mathbf{B}\cdot\nabla\cdot\pi_e >$ no longer contains the driving term $C_b\, E_e(\psi)$ and the bootstrap current is zero. In a collision dominated plasma the bootstrap effect is small and negligeable, however it is interesting to see how this current depends on the geometry of the magnetic surfaces. As already mentioned, configurations with small Pfirsch-Schlüter currents are characterized by small $(\mathbf{V}_o \cdot \frac{\nabla B}{B})$. Therefore, reduction of Pfirsch-Schlüter currents helps to make the bootstrap current small, too.

Since the difference $E_e - E_i$ is proportional to the density gradient the collisional bootstrap current does not depend on the radial electric field. This field has been eliminated using the condition of ambipolarity. Because of the relation $\tau_e << \tau_i$ the toroidal and poloidal ion fluxes are very small compared with the electron fluxes which is a result of the strong viscous damping of the ion velocity. $E_i \approx 0$ implies that the ion pressure gradient is balanced by the radial electric field. This conclusion cannot be drawn for systems with an invariant direction since in this case the determinant $C_pC_t - C_b^2$ is zero, which follows from Eq. (4.8). In symmetric systems ambipolarity and parallel momentum conservation do not uniquely determine X and Y, however the toroidal current can be uniquely calculated in terms of E_e and E_i, as given in Eq. (6.12).

The equation (6.12) for the bootstrap current in a collisional plasma is different from the result given by Shaing, Callen (eq. 79 in Ref.[7]). The reason is that these authors use only the condition $< \mathbf{B}\cdot\nabla\cdot\pi_i >= 0$ to calculate the parallel velocity instead of $< \mathbf{B}\cdot\nabla\cdot\pi_e > + < \mathbf{B}\cdot\nabla\cdot\pi_i >= 0$. The parallel viscous forces are the driving terms of the bootstrap current; these forces, however, are linear in the toroidal fluxes $\Lambda_e(\psi)$ and $\Lambda_i(\psi)$ and therefore a coupled system for these fluxes has to be solved.

In a collisional plasma the second term in the denominator of Eq. (6.12) is negligible and taking into acount the large ion collision time the toroidal current can be simplified to

$$I'(\psi) = C_b 3\frac{kT}{e}\frac{\tau_e}{< B^2 > \eta}(E_e - E_i) \tag{6.13}$$

In a two-fluid plasma Ohm's law is the convenient starting point to derive an equation for the bootstrap current. This is no longer the case in a plasma with several ion species. Here, the parallel momentum balance of each particle species has to be taken as the basis of the analysis. Mulitiplying Eq. (2.1) with $\mathbf{B}$ and taking the surface average yields

$$0 = \sum_k \alpha_{jk} < (\mathbf{V}_j - \mathbf{V}_k)\cdot\mathbf{B} > +3\tau_j P_j < (\mathbf{V}_j \cdot \frac{\nabla B}{B})(\mathbf{B}\cdot\frac{\nabla B}{B}) > \tag{6.14}$$

with the lowest order velocity taken from Eq. (6.6). In explicit form this equation reads

$$0 = \sum_k \alpha_{jk}\{< B^2 > (\Lambda_j(\psi) - \Lambda_k(\psi)) + < \mathbf{B}\cdot\mathbf{V}_o > (E_j(\psi) - E_k(\psi))\} \\ +3\tau_j P_j C_t \Lambda_j(\psi) + 3\tau_j P_j C_b E_j(\psi) \qquad (6.15)$$

With given forces $E_j(\psi)$ this is an inhomogeneous linear system for the toroidal fluxes $\Lambda_j(\psi)$. The system always has a unique solution and the bootstrap current can be found from

$$I'(\psi) = \sum_k e_j N_j \Lambda_j(\psi) \qquad (6.16)$$

The solution $\Lambda_j(\psi)$ minimizes the entropy production or the dissipated power. This power dissipated by the frictional forces is

$$P = \sum_{jk} < \mathbf{V}_j \cdot \mathbf{R}_{jk} > + \sum_j < \mathbf{V}_j \cdot \nabla \cdot \pi_j > \qquad (6.17)$$

or

$$P = \frac{1}{2}\sum_{jk} \alpha_{jk} < (\mathbf{V}_j - \mathbf{V}_k)^2 > + \sum_j 3\tau_j P_j < (\mathbf{V}_j \cdot \frac{\nabla B}{B})^2 > . \qquad (6.18)$$

In order to obtain this form of the dissipated power the following relations have been used:

$$\mathbf{V}_j \times \mathbf{B} = E_j(\psi)\nabla\psi \quad , \quad \nabla \cdot \mathbf{V}_j = 0 \quad , \quad (\mathbf{V}_j \cdot \nabla)\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{V}_j. \qquad (6.19)$$

This frictional and viscous dissipation of energy by the lowest order flow $\mathbf{V}_j$ leads to damping of the diamagnetic and toroidal flow on the magnetic surfaces. The first term describes the frictional effect of the different particle species and the second term, the viscous dissipation, is the magnetic pumping effect. Its role in damping the poloidal flow in tokamaks has been discussed by Hassam and Kulsrud[1]. This magnetic pumping effect is the spatial analogue of the gyro-relaxation effect pointed out by A. Schlüter[2].

Representing the velocity $\mathbf{V}_j$ by Eq. (6.6) yields the power P as a quadratic functional in $E_j(\psi)$ and $\Lambda_j(\psi)$ whose variation with respect to $\Lambda_j(\psi)$ leads to Eq. (6.12). Since P is a positive functional the stationary state is a minimum of the functional and the toroidal fluxes adjust themselves so as to minimize the entropy production rate. Neglecting viscous dissipation would lead to zero bootstrap current since in that case $\Lambda_j(\psi) = \Lambda_k(\psi)$ and $I' = 0$. If the frictional dissiption is neglected, the solution of Eq. (6.12) is

$$\Lambda_j(\psi) = -\frac{C_b}{C_t} E_j(\psi) \qquad (6.20)$$

[1] Ref.[16] A.B. Hassam, R.M. Kuslrud, *Phys. Fluids* **21** (1978) 2271

[2] Ref.[17] A. Schlüter, *Z. f. Naturforsch.***12a** (1957) 822

and the total bootstrap current is

$$I' = -\frac{C_b}{C_t}\sum_j N_j E_j(\psi) = -\frac{C_b}{C_t}\sum_j kT\, N_j'(\psi) \quad (6.21)$$

In this approximation the bootstrap current is independent of the collision frequency and the electric field, however the ansatz for the viscosity only holds in a collisional plasma, therefore, extrapolation to a collisionless plasma has to start from a neoclassical theory of the viscous tensor.

In a tokamak the coefficients C_b and C_t are correlated by $C_t = \iota/V'(\psi)\, C_b$ and Eq. (6.20) is

$$\iota\Lambda_j(\psi) - V'(\psi)E_j(\psi) = 0$$

As shown in Eq. (2.19) the left hand side is the poloidal flux of the particle species j and minimum viscous dissipation is achieved if the poloidal flux is slowed down to zero by the magnetic pumping effect. Including the power dissipated by the bootstrap current leads to a non-zero poloidal rotation in the minimum of entropy production.

VII. The effect of temperature gradients

In the preceeding section we have neglected the inhomogeneity of the plasma temperature which allowed us to investigate the basic relations starting from the momentum balance only. In a plasma with temperature gradients the friction term $\mathbf{R}_{jk}$ is proportional to the thermal flux and the plasma current. The linear relations between frictional forces and thermal fluxes have been derived by Hirshman and Sigmar[1]

$$\mathbf{R}_j = \sum_k l_{11}^{jk}\mathbf{V}_k - \frac{2}{5}l_{12}^{jk}\frac{\mathbf{q}_k^o}{P_k} \quad (7.1)$$

The vector $\mathbf{q}_k^o$ is the lowest order thermal flux of each particle species. The coefficients l_{mn}^{jk} have to be calculated from the collision operator. The presence of the thermal fluxes requires the energy balance to be taken into account and the equation for the energy flux vector $\mathbf{Q}_k$ is

$$\frac{e_j}{m_j}\{\frac{5}{2}P_j\,\mathbf{E} + \mathbf{E} : \pi_j + \mathbf{Q}_j \times \mathbf{B}\} + \mathbf{G}_j - \nabla\cdot\mathbf{r}_j = 0 \quad (7.2)$$

with

$$\mathbf{G}_{jk} = \int m_j\frac{v^2}{2}\mathbf{v}C_{jk}\,d^3v \quad (7.3)$$

$\mathbf{G}_j$ is the heat friction force and is given by

$$\mathbf{G}_j = \frac{T_j}{m_j}(\frac{5}{2}\mathbf{R}_j + \mathbf{F}_j) \quad (7.4)$$

[1] Ref.[7] Eqs. 4.2. We neglect the next order moments $\mathbf{u}_{k2}$

where the vector $\mathbf{F}_{jk}$ is also a linear function of $\mathbf{V}_k$ and $\mathbf{q}_k^o$:

$$\mathbf{F}_j \quad = \quad \sum_k -l_{21}^{jk}\mathbf{V}_k + \frac{2}{5}l_{22}^{jk}\frac{\mathbf{q}_k^o}{P_k} \tag{7.5}$$

The tensor r_{ik} is the energy weighted stress tensor

$$r_{ik} \quad = \quad \int \frac{1}{2}m_j v^2 v_i v_k \, f_j \, d^3\mathbf{v} \tag{7.6}$$

and can also be written in the Chew-Goldberger-Low form

$$r_{ik} \quad = \quad \frac{1}{3}(Tr(r_{ik}))\delta_{ik} + (r_{\|} - r_{\perp})\{b_i b_k - \frac{1}{3}\delta_{ik}\}. \tag{7.7}$$

With these equations of the frictional forces we are in the position to formulate the parallel momentum balance of each particle species

$$< \mathbf{B}\cdot\mathbf{R}_j > \quad - \quad < \mathbf{B}\cdot\nabla\cdot\pi_j > \quad = \quad 0 \tag{7.8}$$

$$< \mathbf{B}\cdot\mathbf{G}_j > \quad - \quad < \mathbf{B}\cdot\nabla\cdot r_j > \quad = \quad 0 \tag{7.9}$$

These equations can be modified to

$$\begin{aligned} \frac{5}{2} < \mathbf{B}\cdot\nabla\cdot\pi_j > - \frac{m_j}{T_j} < \mathbf{B}\cdot\nabla\cdot r_j > + < \mathbf{B}\cdot\mathbf{F}_j > \quad &= \quad 0 \\ - < \mathbf{B}\cdot\nabla\cdot\pi_j > + < \mathbf{B}\cdot\mathbf{R_j} > \quad &= \quad 0 \end{aligned} \tag{7.10}$$

with the friction forces being linear in the fluxes:

$$< \mathbf{B}\cdot\mathbf{R}_j > \quad = \quad \sum_k < B^2 > l_{11}^{jk}\Lambda_k(\psi) - \frac{2}{5}\sum_k l_{12}^{jk}\frac{1}{P_k} < \mathbf{B}\cdot\mathbf{q}_k^o > \tag{7.11}$$

and

$$< \mathbf{B}\cdot\mathbf{F}_j > \quad = - < B^2 > \sum_k l_{21}^{jk}\Lambda_k(\psi) + \sum_k \frac{2}{5}l_{22}^{jk}\frac{1}{P_j} < \mathbf{B}\cdot\mathbf{q}_k^o > \tag{7.12}$$

These two equations (7.8) and (7.9) represent an inhomogeneous system for the fluxes $\Lambda_k(\psi)$ and $< \mathbf{B}\cdot\mathbf{q}_k >$. However, the system is singular since the momentum conservation of Coulomb collisions requires

$$\sum_{jk} < \mathbf{B}\cdot\mathbf{R}_{jk} > \quad = \quad 0$$

Following the notation of Hirshman and Sigmar the term $\sum l_{11}^{jk}\Lambda_k(\psi)$ can be written

$$\sum_k l_{11}^{jk}\Lambda_k(\psi) \quad = \quad \sum_k m_j N_j \nu_{jk} M_{jk}^{00}\left(\Lambda_j(\psi) - \Lambda_k(\psi)\right) \tag{7.13}$$

and

$$\sum_k l_{21}^{jk} \Lambda_k(\psi) = \sum_k m_j N_j \nu_{jk} M_{jk}^{10} \left(\Lambda_j(\psi) - \Lambda_k(\psi)\right) \tag{7.14}$$

where M_{jk}^{00} and M_{jk}^{10} are symmetric matrices. In the parallel energy balance equation the term with $\Lambda_k(\psi)$ can also be modified into the same form and the resulting equations only contain the relative toroidal fluxes $\Lambda_j(\psi) - \Lambda_k(\psi)$. This implies that one of the toroidal fluxes remains undetermined after solving the system. The driving terms of parallel fluxes and parallel heat flow are the parallel viscosity and the parallel energy weighted viscosity.

In the following we restrict the analysis to a two-fluid plasma, since in that case the resulting system of equations can still be solved without numerical effort. In a two-fluid plasma with single-charged ions only the difference $\Lambda_e(\psi) - \Lambda_i(\psi)$ occurs which is proportional to the toroidal plasma current. In this case the equations (7.10) - (7.12) are

$$\begin{aligned}
< B^2 > \{ (l_{11}^{ee}\Lambda_e(\psi) + l_{11}^{ei}\Lambda_i(\psi)) - \frac{2}{5} l_{12}^{ee} X_e - \frac{2}{5} l_{12}^{ei} X_i \} &= A_e \\
< B^2 > \{ (l_{11}^{ie}\Lambda_e(\psi) + l_{11}^{ii}\Lambda_i(\psi)) - \frac{2}{5} l_{12}^{ie} X_e - \frac{2}{5} l_{12}^{ii} X_i \} &= A_i \\
- < B^2 > \{ (l_{21}^{ee}\Lambda_e(\psi) + l_{21}^{ei}\Lambda_i(\psi)) - \frac{2}{5} l_{22}^{ee} X_e - \frac{2}{5} l_{22}^{ei} X_i \} &= B_e \\
- < B^2 > \{ (l_{21}^{ie}\Lambda_e(\psi) + l_{21}^{ii}\Lambda_i(\psi)) - \frac{2}{5} l_{22}^{ie} X_e - \frac{2}{5} l_{22}^{ii} X_i \} &= B_i .
\end{aligned} \tag{7.15}$$

Here we have introduced the following notation

$$A_e = < \mathbf{B} \cdot \nabla \cdot \pi_e > \quad ; \quad A_i = < \mathbf{B} \cdot \nabla \cdot \pi_i > \tag{7.16}$$

and

$$\begin{aligned}
B_e &= < \mathbf{B} \cdot \nabla \cdot \Theta_e > = < \frac{m_e}{T_e} \mathbf{B} \cdot \nabla \cdot r_e > - \frac{5}{2} < \mathbf{B} \cdot \nabla \cdot \pi_e > \\
B_i &= < \mathbf{B} \cdot \nabla \cdot \Theta_i > = < \frac{m_i}{T_i} \mathbf{B} \cdot \nabla \cdot r_i > - \frac{5}{2} < \mathbf{B} \cdot \nabla \cdot \pi_i >
\end{aligned} \tag{7.17}$$

The terms X_e, X_i are defined by

$$X_e < B^2 > = < \mathbf{B} \cdot \mathbf{q}_e^o > \qquad X_i < B^2 > = < \mathbf{B} \cdot \mathbf{q}_i^o > \tag{7.18}$$

The equations (7.15) represent an inhomogeneous system for the unknown terms X_e, X_i, $\Lambda_e(\psi)$, $\Lambda_i(\psi)$. After eliminating the parallel thermal fluxes X_e, X_i the parallel fluxes $\Lambda_e(\psi)$ and $\Lambda_i(\psi)$ can be written in terms of the parallel viscous forces A_e, A_i, B_e, B_i and using the symmetry relations of the coefficients l_{jk}^{ei} and $l_{22}^{ii} >> l_{22}^{ee}$ the bootstrap current is

$$eN\eta < B^2 > I'(\psi) = < \mathbf{B} \cdot \nabla \cdot \pi_e > + \frac{l_{12}^{ee}}{l_{22}^{ee}} < \mathbf{B} \cdot \nabla \cdot \Theta_e > - \frac{l_{22}^{ei}}{l_{22}^{ii}} < \mathbf{B} \cdot \nabla \cdot \Theta_i > \tag{7.19}$$

The Spitzer resistivity is

$$N^2 e^2 \eta = l_{11}^{ee} - \frac{(l_{12}^{ee})^2}{l_{22}^{ee}} \tag{7.20}$$

The last term proportional to the ion heat viscosity is very small, since $l_{22}^{ei}/l_{22}^{ii} \approx (\frac{m_e}{m_i})^{3/2}$. Although the parallel thermal flow of the ions can be much larger than the thermal flow of the electrons, the ion parallel heat viscosity does not contribute to the bootstrap current. Therefore, the ion term will be neglected in the following. The tensor Θ can be also written in the C.G.L-form

$$\Theta_{ik} = \frac{1}{3}(Tr\Theta_{ik})\delta_{ik} + (\Theta_{\parallel} - \Theta_{\perp})\{b_i b_k - \frac{1}{3}\delta_{ik}\}. \tag{7.21}$$

The equations (7.19) are valid for arbitrary asymmetric or symmetric configurations, the remaining task is now to relate the parallel viscous forces to the poloidal viscous forces and thus represent the bootstrap current in terms of the neoclassical radial fluxes. In symmetric systems this can be easily achieved by using the proportionality between parallel and poloidal derivatives of B:

$$< (\Theta_{\parallel} - \Theta_{\perp})\,(\mathbf{B} \cdot \frac{\nabla B}{B}) > = \frac{C_2}{C_3}(\iota - \gamma_\omega) \; < (\Theta_{\parallel} - \Theta_{\perp})(\mathbf{V}_o \cdot \frac{\nabla B}{B}) > \tag{7.22}$$

which correlates the neoclassical bootstrap current to the neoclassical particle flux and the energy fluxes of electrons and ions. Taking the poloidal average of the energy flux equation (7.2) yields the total energy flux

$$\frac{1}{V'(\psi)} e_j Q_j = < m_j\, \mathbf{G}_j \cdot \mathbf{V}_o > - m_j < \mathbf{V}_o \cdot \nabla \cdot r_j > \tag{7.23}$$

or together with the particle balance

$$\frac{1}{V'(\psi)} e_j \Gamma_j = < \mathbf{R_j} \cdot \mathbf{V}_o > \; - \; < \mathbf{V}_o \cdot \nabla \cdot \pi_j >$$

we obtain the thermal flux $q_j = Q_j - 5/2 T_j \Gamma_j$

$$\frac{1}{V'(\psi)} e_j q_j = < \mathbf{V}_o \cdot \mathbf{F}_j > \; - \; < \mathbf{V}_o \cdot \nabla \cdot \Theta_j > . \tag{7.24}$$

The first term on the right hand side is the Pfirsch-Schlüter thermal flux and the second term is the neoclassical thermal flux.

$$\frac{e_j}{V'(\psi)\, T_j} q_{j,neo} = - < \mathbf{V}_o \cdot \nabla \cdot \Theta_j > \tag{7.25}$$

and together with the neoclassical particle flux

$$\frac{e_j}{V'(\psi)} \Gamma_{j,neo} = - < \mathbf{V}_o \cdot \nabla \cdot \pi_j > \tag{7.26}$$

we can write the bootstrap current

$$\eta < B^2 > I'(\psi) = \frac{C_2}{C_3\, V'(\psi)}(\iota - \gamma_\omega)\{\frac{\Gamma_{neo}}{N} + \frac{l_{12}^e}{l_{22}^e}\frac{q_{e,neo}}{N T_e}\}. \tag{7.27}$$

This equation is the desired relation between bootstrap current and the neoclassical radial fluxes of the plasma. It can be derived without solving a kinetic equation and holds for all regimes of collisionality. Again, the effective loop voltage driving the bootstrap current can be defined by

$$U_{eff} = \frac{\bar{B}}{2\pi R}(\iota - \gamma_\omega)\{\frac{\Gamma_{neo}}{N} + \frac{l^e_{12}}{l^e_{22}}\frac{q_{e,neo}}{NT_e}\}. \tag{7.28}$$

or in terms of the energy confinement times which are defined by

$$q = \frac{3}{2}\frac{\overline{NTV}}{\tau_E}$$

$$U_{eff} = \bar{B}\pi r^2(\iota - \gamma_\omega)\{\frac{\bar{N}}{N\tau_p} + 0.48\frac{\overline{NT_e}}{NT_e\tau_{E,e}}\}. \tag{7.29}$$

VIII. Effect on the rotational transform

The bootstrap current modifies the rotational transform in a stellarator or tokamak configuration. The size of this variation depends on the plasma pressure and may cause serious problems during the rise phase of a discharge when the rotational transform passes through a low order rational value at the plasma boundary. In the following we concentrate on symmetric configurations and compute the bootstrap current and the corresponding variation of the rotational transform explicitly. In order to do so we introduce the following approximations:

$$C_3 = V'(\psi) \approx \frac{L}{B_o} \quad , \quad < B^2 > \approx B_o^2 \tag{8.1}$$

where L is the length of the magnetic axis and B_o is defined by this relation. Furthermore we define the effective major radius by the relation $L = 2\pi R$. The toroidal flux ψ within a magnetic surface defines the average minor radius of the magnetic surface by

$$\psi = B_1 \pi r^2 \tag{8.2}$$

B_1 is a reference field and can be taken equal to B_o. The volume of the magnetic surface is

$$V \approx L\pi^2 r^2 = 2\pi r^2 R \tag{8.3}$$

and the area of the magnetic surface

$$F(\psi) = C_F(r)\, 4\pi^2 Rr \tag{8.4}$$

C_F is a factor of order one and depends on the geometry of the magnetic surface.

The neoclassical fluxes generally are given as linear functions of the gradients

$$\begin{pmatrix} \Gamma \\ q_e \\ q_i \end{pmatrix} = F\,(D_{ik}) \begin{pmatrix} \frac{N'}{N} \\ \frac{T_e'}{T_e} \\ \frac{T_i'}{T_i} \end{pmatrix} \tag{8.5}$$

with D_{ik} being the matrix of transport coefficients. Using these definitions and approximations the bootstrap current is written in terms of the density and temperature gradients

$$I' = \frac{C_F(\iota - \gamma_\omega)}{\eta R}\{r \sum_{i,k} C_i D_{ik} A_k\} \tag{8.6}$$

where the vectors **C** and **A** are defined by

$$\mathbf{C} = \begin{pmatrix} 1 \\ l_{12}^e \\ l_{22}^e \\ l_{12}^e\, l_{22}^e \\ l_{22}^e\, l_{22}^i \end{pmatrix} \qquad \mathbf{A} = \begin{pmatrix} \frac{N'}{N} \\ \frac{T_e'}{T_e} \\ \frac{T_i'}{T_i} \end{pmatrix} \tag{8.7}$$

With $d\psi = 2B_o\pi r\, dr = 2B_o\pi a^2 x\, dx$ ($x = r/a$ and $a =$ plasma radius) the integrated bootstrap current is

$$I(a) = I_o \int_0^1 (\iota - \gamma_\omega)\frac{x^2}{\eta} \sum_{ik} C_i D_{ik} A_k\, dx. \tag{8.8}$$

The reference current I_o is given by $2\pi a^2 B_o/\mu_o R$ and the gradients have to be taken with respect to the dimensionless variable x. The reference current changes the rotational transform by $\delta\iota = 1$,which can be seen from

$$\delta\iota_o = \frac{R}{a}\frac{\mu_o I_o}{B_o 2\pi a} = 1 \tag{8.9}$$

and consequently

$$\delta\iota = \int_0^1 (\iota - \gamma_\omega)\frac{x^2}{\eta} \sum C_i D_{ik} A_k\, dx \tag{8.10}$$

If the transport coefficients D_{ik} are normalized to the diffusion coefficient D_e of the electrons this equation becomes

$$\delta\iota = \int_0^1 (\iota - \gamma_\omega)\frac{x^2}{\eta} D_e \sum C_i d_{ik} A_k\, dx. \tag{8.11}$$

The matrix elements d_{ik} are numerical factors.

In the following we consider a simple case with all plasma profiles being similar $A_k = N'/N$. In this case the factor $C_o = \sum C_i d_{ik}$ is a numerical factor between 1 and 3.

Tokamak case

In the plateau regime the bootstrap current increases with η^{-1} since the diffusion coefficient D_e is independent of the collision frequency. The maximum bootstrap current arises in the banana regime where the current is independent of the collision frequency ν since both terms - D_e and η - are proportional to ν. With

$$D_e = \nu \frac{\rho_e^2}{\iota^2}(\frac{R}{r})^{3/2} \tag{8.12}$$

and

$$\eta = 0.5 \frac{m_e \nu}{N e^2} \quad ; \quad \gamma_\omega = 0$$

we obtain

$$\frac{D_e}{\eta} = \frac{4N\, kT_e}{B^2} \frac{1}{\iota^2}(\frac{R}{r})^{3/2} \tag{8.13}$$

and the bootstrap current in the tokamak is

$$I(a) = 2I_o\,(\frac{R}{a})^{3/2} \int_0^1 x^{1/2} \frac{\beta_e(x)}{\iota(x)}\, C_o\, \frac{N'}{N}\, dx \tag{8.14}$$

The neoclassical fluxes in the banana regime can be found in a paper by Rosenbluth, Hazeltine and Hinton[1]

$$\begin{aligned} \Gamma_e &= F(r)\, D_e N \left\{-1.12(1+\frac{T_i}{T_e})\frac{N'}{N} + 0.43\frac{T_e'}{T_e} + 0.19\frac{T_i'}{T_e}\right\} \\ q_e &= F(r)\, D_e N T_e \left\{1.53(1+\frac{T_i}{T_e})\frac{N'}{N} - 1.80\frac{T_e'}{T_e} - 0.27\frac{T_i'}{T_e}\right\} \\ q_i &= -F(r)\, D_i N T_i\, 0.68 \frac{T_i'}{T_i} \end{aligned} \tag{8.15}$$

$F(r) = 4\pi^2\, rR$ is the area of the magnetic surface. From these equations we find

$$\Gamma_e + 0.32\frac{q_e}{T_e} = F(r)\, D_e N\{-0.63(1+\frac{T_i}{T_e})\frac{N'}{N} - 0.15\frac{T_e'}{T_e} + 0.1\frac{T_i'}{T_e}\} \tag{8.16}$$

and with the approximation $T_e \approx T_i$, $N'/N = T'/T$ the result is reduced to

$$\Gamma_e + 0.32\frac{q_e}{T_e} \approx -1.3\, F(r)\, D_e N\, \frac{N'}{N} \tag{8.17}$$

which leads to the numerical coefficient $C_o \approx 1.3$. With $\beta_e \approx \beta_i = 0.5\,\beta$ we can write

$$I(a) = 1.3\, I_o\,(\frac{R}{a})^{3/2} \int_0^1 x^{1/2} \frac{\beta(x)}{\iota(x)} \frac{N'}{N}\, dx \tag{8.18}$$

[1]Ref.[18] M.N. Rosenbluth, R.D. Hazeltine and F.L. Hinton *Phys. Fluids* **15** (1972), 116

The ansatz $N \propto (1-x^2)^\alpha$ yields

$$I(a) = 1.3 I_o \beta(0) (\frac{R}{a})^{3/2} G(\alpha) \tag{8.19}$$

with

$$G(\alpha) = 2\alpha \int_0^1 x^{3/2}(1-x^2)^{2\alpha-1} \frac{1}{\iota} dx$$

The numerical profile factor $G(\alpha)$ is nearly independent of α and approximately equal to 0.3 if we approximate the rotational transform by 1 and therefore the bootstrap current in tokamaks is roughly

$$I(a) = 0.4 I_o \beta(0) (\frac{R}{a})^{3/2} \tag{8.20}$$

and the change of the rotational transform

$$\delta\iota \approx 0.4 \beta(0) (\frac{R}{a})^{3/2} \tag{8.21}$$

In a tokamak with aspect ratio 4 and a maximum β of 0.1 the variation of the rotational transform is $\delta\iota = 0.32$. Since $\frac{1}{\iota}$ varies between 1 and 3 this number can be slightly larger.

Helical configuration

In a helically invariant configuration or in quasi-helical systems the banana width is smaller than in the tokamak system. The banana width is roughly

$$\Delta \approx \frac{\rho}{(\iota - \gamma_\omega)} (\frac{R}{r}) \sqrt{\epsilon} \tag{8.22}$$

and the diffusion coefficient

$$D_e = \nu \frac{\rho_e^2}{(\iota - \gamma_\omega)^2} (\frac{R}{r})^2 \sqrt{\epsilon}$$

$\sqrt{\epsilon}$ is the number of trapped particles. Boozer[1] has shown that axisymmetric and helically symmetric configurations are isomorphic with respect to neoclassical transport and therefore it may be justified to take the same numerical factors d_{ik} as in the tokamak. This leads to the following equation for the bootstrap current in helical systems:

$$I(a) = 1.3 I_o \beta(0) (\frac{R}{a})^2 \sqrt{\epsilon} G(\alpha) \tag{8.23}$$

where the profile factor $G(\alpha)$ is given by

$$G(\alpha) = \frac{2\alpha}{(\iota - \gamma_\omega)} \int_0^1 x^{3/2}(1-x^2)^{2\alpha-1} dx \approx \frac{0.3}{(\iota - \gamma_\omega)} \tag{8.24}$$

[1] Ref.[19] A.H. Boozer, *Phys. Fluids* **26** (1983) 496

Here we have assumed that ϵ is proportional to x. In comparing the helical system with the axisymmetric tokamak we find that the bootstrap current in helical systems is reduced because of the smaller banana size. Since on the other hand the aspect ratio in quasi-helical systems is larger than in tokamaks this leads to an increase of the bootstrap current so that eventually the current in quasi-helical configurations is only slightly smaller than in tokamaks.

In the paper of Pytte and Boozer the bootstrap current in helically invariant systems is directly calculated from the kinetic theory[1] and the following current density is found

$$j_{||} = 1.46\sqrt{\epsilon}\frac{c}{B_o \alpha r}\, p' \tag{8.22}$$

from which the total current can be calculated

$$I(a) = I_o\, 0.73\, \frac{1}{\gamma_\omega}(\frac{R}{a})^2\sqrt{\epsilon}\int\limits_0^1 x^{1/2}\beta'\, dx \tag{8.23}$$

In an $l = 1$ helical system the slope γ_ω is equal to the number of field periods. This relation is nearly the same as derived above except that the factor $(\iota - \gamma_\omega)$ is replaced by γ_ω.

[1] Ref.[15]

IX. Neoclassical Plasma

In the preceeding chapters a collision dominated plasma has been investigated. Many relations, however, are also valid in the collisionless regime, especially the flux-friction-relations described in chapter III, where only the general form of the Chew-Goldberger-Low pressure tensor has been used. In a hot plasma the mean free path is long compared with the dimensions of the plasma and therefore the plasma pressure is dominated by particle drifts rather than by collisions. The anisotropic pressure

$$(p_{\|} - p_{\perp})_j = m_j \int g_j \, (v_{\|}^2 - \frac{1}{2} v_{\perp}^2) \, d^3 v \tag{9.1}$$

has to be calculated from the distribution function g_j which is the solution of the kinetic equation:[1]

$$v_{\|} \mathbf{b} \cdot \nabla g_j + \mathbf{v_D} \cdot \nabla g_j - C(g_j) = -\{ \mathbf{v_D} \cdot \nabla \psi f_o{}'(\psi) + f_o v_{\|} \mathbf{b} \cdot \nabla [\frac{2 v_{\|}}{v_{th}^2} U_{\|}] \} \tag{9.2}$$

or in shorter notation

$$L g_j = h_j \, . \tag{9.3}$$

$L = v_{\|} \mathbf{b} \cdot \nabla + \mathbf{v_D} \cdot \nabla - C$ is the operator on the left hand side and $C(g_j)$ is the test particle collision operator. $U_{\|}$ is the parallel component of the macroscopic flow $\mathbf{V_j}$ and v_{th} the thermal velocity. The guiding center drift velocity $\mathbf{v_D}$ is

$$\mathbf{v_D} = \frac{v_{\|}}{B} \frac{m_j}{e_j} \nabla \times v_{\|} \mathbf{b} \tag{9.4}$$

with $v_{\|} = \sqrt{v^2 - \mu B + e\Phi} = v_{\|}(E, \mu, B)$. $\mathbf{v_D}$ is the sum of the magnetic drift and the $\mathbf{E} \times \mathbf{B}$-drift in the magnetic surface. On the left hand side of the kinetic equation usually the radial component of the drift velocity is neglected, thus making the operator L operate on two surface variables and two velocity space variables.

In a collision dominated plasma the averaged viscous forces are shown to be linear functions of $E_j(\psi)$ and $\Lambda_j(\psi)$. In order to derive the equivalent relation for a neoclassical plasma the right hand side of the kinetic equation has to be modified to a linear relationship with $E_j(\psi)$ and $\Lambda_j(\psi)$. To make the analysis simple, an isothermal plasma with constant

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[1] This form of the drift kinetic equation results from the transformation of the kinetic equation on a coordinate system moving with the macroscopic velocity $U_{\|}$. In Eq. (9.2) the velocity $v_{\|}$ is the random velocity. For the solution of the kinetic equation the following ansatz

$$f = f_o + 2 \frac{v_{\|} U_{\|}}{v_{th}^2} f_o + g.$$

is made. The distribution g is the next order correction to the shifted Maxwellian and has to be calculated from Eq. (9.2). (see paper of M. Coronado, H. Wobig *Phys. Fluids* **29**, (1986) 527 and also K.C. Shaing *Phys. Fluids* **31**, (1988) 8).

temperature is considered first. The derivative of the lowest order Maxwellian distribution function is

$$f_o'(\psi) = f_o\left(\frac{N'}{N} + \frac{e_j \Phi'}{kT}\right) = \frac{e_j}{kT} f_o \, E_j(\psi) \,. \tag{9.5}$$

The normal component of the drift velocity can be written

$$\mathbf{v_D} \cdot \nabla\psi = \frac{m_j}{e_j} \frac{v_\parallel}{B} \nabla \cdot \left(\frac{v_\parallel}{B} \mathbf{B} \times \nabla\psi\right) \tag{9.6}$$

and the right hand side of the kinetic equation is modified to

$$h_j = -\frac{m_j}{kT} f_o \frac{v_\parallel}{B} \nabla \cdot [v_\parallel B (E_j(\psi) \frac{\mathbf{B} \times \nabla\psi}{B^2} + U_\parallel \mathbf{b})] \,. \tag{9.7}$$

The term in the brackets is the lowest order fluid velocity

$$\mathbf{V}_j \; = \; E_j(\psi) \frac{\mathbf{B} \times \nabla\psi}{B^2} + U_\parallel \mathbf{b}$$

Using $\nabla \cdot \mathbf{V}_j = 0$ and

$$\nabla v_\parallel B = \frac{(v_\parallel^2 - \frac{1}{2} v_\perp^2)}{v_\parallel} \nabla B$$

the driving term in the kinetic equation can be written

$$h_j = -\frac{m_j}{kT} f_o (v_\parallel^2 - \frac{1}{2} v_\perp^2) \mathbf{V}_j \cdot \frac{\nabla B}{B} . \tag{9.8}$$

With $\mathbf{V}_j$ given by eq. 1.29 and the definitions

$$\begin{aligned} w_1 &= (v_\parallel^2 - \frac{1}{2} v_\perp^2)(\mathbf{V}_o \cdot \frac{\nabla B}{B}) \\ w_2 &= (v_\parallel^2 - \frac{1}{2} v_\perp^2)(\mathbf{B} \cdot \frac{\nabla B}{B}) \end{aligned} \tag{9.9}$$

the kinetic equation is

$$L \, g_j = -\frac{m_j}{kT} f_o \{ w_1 E_j(\psi) + w_2 \Lambda_j(\psi) \} \tag{9.10}$$

The solution of this kinetic equation is the main task of neoclassical theory, a general solution cannot be given. Since the operator L has no derivatives with respect to the radial coordinate ψ the solution of the kinetic equation can be written formally

$$g_j = -\frac{m_j}{kT} \{ E_j(\psi) \, L^{-1} \, f_o w_1 + \Lambda_j(\psi) \, L^{-1} \, f_o w_2 \} \tag{9.11}$$

L^{-1} is the inverse of the operator L. In a collision dominated plasma L^{-1} is mainly the inverse of the collision operator C; in the long mean free path regime integration over particle orbits dominates. From the distribution function g_j the anisotropic part of the pressure $(p_{\|} - p_{\perp})_j$ is calculated

$$(p_{\|} - p_{\perp})_j = -\frac{m_j^2}{kT}\{E_j(\psi)\int (v_{\|}^2 - \frac{1}{2}v_{\perp}^2)\, L^{-1}\, f_o w_1 d^3\, v \\ +\Lambda_j(\psi)\int (v_{\|}^2 - \frac{1}{2}v_{\perp}^2)\, L^{-1}\, f_o w_2 d^3 v\} \tag{9.12}$$

and the surface averaged parallel and perpendicular viscosities (Eqs. (3.18)) are

$$\begin{aligned} - < \mathbf{B}\cdot\nabla\cdot\pi_j > &= \mu_j^{21} E_j(\psi) + \mu_j^{22}\Lambda_j(\psi) \\ - < \mathbf{V}_o\cdot\nabla\cdot\pi_j > &= \mu_j^{11} E_j(\psi) + \mu_j^{12}\Lambda_j(\psi) \end{aligned} \tag{9.13}$$

where the coefficients μ_j^{ik} are defined by

$$-\mu_j^{ik} = \frac{m_j^2}{kT} < \int w_i\, L^{-1}\, f_o w_k\, d^3 v > \quad ; \quad i,k = 1,2 \tag{9.14}$$

These coefficients are the neoclassical transport coefficients for arbitrary collisionality. The coefficients μ^{11} are the radial transport coefficients since they correlate the neoclassical fluxes to the gradients $E_j(\psi)$. μ^{12} is the Ware pinch coefficient and μ^{21} the bootstrap coefficient. Because of the symmetry of these effects the matrix μ^{ik} is symmetric. The coefficients μ^{22} describe the damping of the lowest order flow by the magnetic pumping effect.

Having written the surface averaged viscous forces in terms of the macroscopic quantities $E_j(\psi)$ and $\Lambda_j(\psi)$ the flux-friction relations are

$$\sum_k \alpha_{jk} < B^2 > (\Lambda_j(\psi) - \Lambda_k(\psi)) + \mu_j^{22}\Lambda_j(\psi) \\ + \mu_j^{21} E_j(\psi) \; = \; - < \mathbf{B}\cdot\xi_j > \tag{9.15}$$

$$\frac{1}{V'(\psi)} e_j\,\Gamma_j = \sum_k \alpha_{jk} < \mathbf{V}_o{}^2 > (E_j(\psi) - E_k(\psi)) \; + \; \mu_j^{11} E_j(\psi) \\ + \mu_j^{12}\Lambda_j(\psi) \; + \; < \mathbf{V}_o\cdot\xi_j > \tag{9.16}$$

These flux - friction relations for a particle species j are obtained from the momentum balance equations (1.1) by multiplying with $\mathbf{V}_o$ and $\mathbf{B}$ and averaging over the magnetic surface. α_{ik} is the friction coefficient between different particle species. Inertial forces and the external momentum sources imposed by the heating mechanisms are denoted with ξ_j. Summing up over particle species yields the equations of parallel momentum balance and the condition of ambipolarity

$$\sum_j \mu_j^{22}\Lambda_j(\psi) + \sum_j \mu_j^{21} E_j(\psi) + \sum_j < \mathbf{B}\cdot\xi_j > = 0 \tag{9.17}$$

$$\sum_j \mu_j^{11} E_j(\psi) + \sum_j \mu_j^{12} \Lambda_j(\psi) + \sum_j < \mathbf{V}_o \cdot \xi_j > = 0 \tag{9.18}$$

These equations complete the system of flux-friction-relations; they are similar to those obtained in a collision-dominated plasma, only the transport coefficients have to be replaced by the appropriate neoclassical coefficients.

Eqs. (9.15) and (9.16) are the basic equations which correlate the toroidal flows to the gradients and particle fluxes, the effect of external momentum sources is included by the terms with ξ_j.

Two-fluid plasma

In the following we consider a two-fluid plasma without external momentum sources and eliminate the gradients $E_j(\psi)$ from the equations in order to represent the toroidal fluxes $\Lambda_j(\psi)$ by the radial particle fluxes. These four equations can be studied conveniently if a matrix notation is being used. For this purpose the following definitions are introduced:

$$L_{11} = \begin{pmatrix} \mu_e^{11} & 0 \\ 0 & \mu_i^{11} \end{pmatrix} \qquad L_{22} = \begin{pmatrix} \mu_e^{22} & 0 \\ 0 & \mu_i^{22} \end{pmatrix} \tag{9.19}$$

$$L_{21} = \begin{pmatrix} \mu_e^{21} & 0 \\ 0 & \mu_i^{21} \end{pmatrix} \qquad L_{12} = \begin{pmatrix} \mu_e^{12} & 0 \\ 0 & \mu_i^{12} \end{pmatrix} \tag{9.20}$$

and

$$\mathbf{S} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \; ; \; \mathbf{I} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; ; \; \mathbf{N} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{9.21}$$

Furthermore, two vectors are defined by

$$\mathbf{e}_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \; ; \; \mathbf{e}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

with the following properties

$$\mathbf{N}\,\mathbf{e}_+ = \mathbf{e}_+, \quad \mathbf{1}\,\mathbf{e}_+ = \mathbf{e}_+, \quad \mathbf{N}\,\mathbf{e}_+ = -\mathbf{e}_+, \quad \mathbf{1}\,\mathbf{e}_- = \mathbf{e}_-$$
$$\mathbf{I}\,\mathbf{e}_+ = 2\mathbf{e}_+, \quad \mathbf{I}\,\mathbf{e}_- = 0, \quad \mathbf{S}\,\mathbf{e}_+ = 0, \quad \mathbf{S}\,\mathbf{e}_- = 2\mathbf{e}_- \tag{9.22}$$

and the orthogonality $\mathbf{e}_+ \cdot \mathbf{e}_- = 0$. The matrix **S** has the following properties: Let $\mathbf{x} = \{x_1, x_2\}$ be an arbitrary vector and A an arbitrary two-by-two diagonal matrix, then the following relations hold

$$\mathbf{S}\,\mathbf{x} = (x_1 - x_2)\,\mathbf{e}_- \qquad \mathbf{S}\,A\,\mathbf{e}_- = (Tr\,A)\,\mathbf{e}_-$$

These relations are very useful for writing the basic system (9.15) and (9.16) in a shorter form which after defining the vectors $\mathbf{E} = \{E_e, E_i\}$ and $\mathbf{\Lambda} = \{\Lambda_e(\psi), \Lambda_i(\psi)\}$ is reduced to

$$\begin{aligned} 0 &= L_{22}\,\mathbf{\Lambda} + L_{21}\,\mathbf{E} + <B^2> \alpha_{ei}\,\mathbf{S\Lambda} \\ \frac{e\,\Gamma}{V'(\psi)}\,\mathbf{e}_- &= L_{11}\mathbf{E} + L_{12}\,\mathbf{\Lambda} + <V_o^2> \alpha_{ei}\mathbf{SE} \end{aligned} \tag{9.23}$$

The last term in the second equation is the Pfirsch-Schlüter diffusion and the first two terms describe the neoclassical effects. The Pfirsch-Schlüter flux is intrinsically ambipolar and depends only on the difference $E_e - E_i$: $\mathbf{SE} = (E_e - E_i)\mathbf{e}_-$ and after defining the neoclassical particle flux by

$$\frac{e\,\Gamma_{neo}}{V'(\psi)}\mathbf{e}_- =: \frac{e\Gamma}{V'(\psi)}\mathbf{e}_- \; - <V_o^2> \alpha_{ei}(E_e - E_i)\mathbf{e}_-$$

the equations are

$$\begin{aligned} 0 &= L_{22}\,\mathbf{\Lambda} + L_{21}\,\mathbf{E} + <B^2> \alpha_{ei}\,\mathbf{S\Lambda} \\ \frac{e\,\Gamma_{neo}}{V'(\psi)}\,\mathbf{e}_- &= L_{11}\mathbf{E} + L_{12}\,\mathbf{\Lambda} \end{aligned} \tag{9.24}$$

These equations allow one to compute the toroidal fluxes $\Lambda_e(\psi), \Lambda_i(\psi)$ as functions of Γ_{neo}. Eliminating the vector $\mathbf{E}$ from these equations provides the desired relation between the toroidal fluxes and the radial particle flux.

$$-\frac{e\Gamma_{neo}}{V'(\psi)}\,L_{21}\,\mathbf{e}_- = <B^2> \alpha_{ei}\,L_{11}\,\mathbf{S\Lambda} + D\mathbf{\Lambda} \tag{9.25}$$

where D is the following diagonal matrix $D = L_{22}L_{11} - L_{21}L_{12}$, or explicitly

$$D = \begin{pmatrix} \mu_e^{22}\mu_e^{11} - \mu_e^{21}\mu_e^{12} & 0 \\ 0 & \mu_i^{22}\mu_i^{11} - \mu_i^{12}\mu_i^{21} \end{pmatrix} =: \begin{pmatrix} d_e & 0 \\ 0 & d_i \end{pmatrix} \tag{9.26}.$$

The fluxes $\mathbf{\Lambda}$ can be uniquely determined if the matrix on the right hand side of Eq. (9.25) $< B^2 > \alpha_{ei}L_{11}\mathbf{S} + D$ is non-singular. This, however, is not always the case since in symmetric configurations the matrix D is identically zero and $\mathbf{S}$ is a singular matrix. As has been shown in chapter IV Eq. (4.8), the derivatives of B in the parallel and poloidal direction are proportional to each other in symmetric systems

$$\mathbf{B}\cdot\nabla B = \frac{C_2}{C_3}(\iota - \gamma_\omega)\mathbf{V}_o\cdot\nabla B\,.$$

and therefore all coefficients μ^{ik}

$$\mu^{22} = M^2 \mu^{11}, \quad \mu^{12} = \mu^{21} = M \mu^{11}.$$

With $M = C_2/C_3\,(t - \gamma_\omega)$ we obtain

$$\mu^{22}\mu^{11} - \mu^{12}\mu^{21} = 0.$$

Therefore, in tokamaks and helically invariant systems the matrix D is singular and the relation between neoclassical particle flux and bootstrap current is reduced to

$$\frac{e\,\Gamma_{neo}}{V'(\psi)} L_{21}\,\mathbf{e}_- = < B^2 > \alpha_{ei}\, L_{11}\,\mathbf{S\Lambda} \tag{9.27}$$

or by multiplying with $\mathbf{e}_- \cdot \mathbf{S}$

$$-\frac{e\,\Gamma_{neo}}{V'(\psi)} Tr\, L_{21} = < B^2 > \alpha_{ei}\,(\Lambda_e(\psi) - \Lambda_i(\psi))\, Tr\, L_{11} \tag{9.28}$$

which is equivalent to

$$< B^2 > \alpha_{ei}\,(\Lambda_e(\psi) - \Lambda_i(\psi)) = -\frac{C_2}{C_3}\,(t - \gamma_\omega)\,\frac{e\,\Gamma_{neo}}{V'(\psi)}. \tag{9.29}$$

This is the same equation as found in chapter IV Eq. (4.11). In this symmetric case only the difference $\Lambda_e(\psi) - \Lambda_i(\psi)$ can be calculated from the particle flux since $S\Lambda = (\Lambda_e(\psi) - \Lambda_i(\psi))\mathbf{e}_-$ depends only on the difference of the toroidal fluxes. This implies that one of the toroidal fluxes remains undetermined in symmetric systems. The plasma can freely move in the invariant direction, which implies that standard neoclassical theory fails to predict this rotation uniquely. The reason is that in standard neoclassical theory shear viscosity and gyroviscosity are neglected. These terms would carry momentum to the wall and thus introduce a slowing down mechanism. In the general non-symmetric case the bulk viscosity leads to a damping of the macroscopic flow and the ambiguity of the integral fluxes is removed.

In non-axisymmetric configurations the matrix D is non-singular and Eq. (9.25) can be uniquely inverted with respect $\Lambda_e(\psi)$ and $\Lambda_i(\psi)$. The difference $\Lambda_e(\psi) - \Lambda_i(\psi)$ is given by

$$\Lambda_e(\psi) - \Lambda_i(\psi) = -\frac{Tr\,(D^{-1}L_{21})}{< B^2 > \alpha_{ei}\, Tr\,(D^{-1}L_{11}) + 1}\,\frac{e\,\Gamma_{neo}}{V'(\psi)} \tag{9.30}$$

or, taking into account the explicit form of the matrices

$$\Lambda_e(\psi) - \Lambda_i(\psi) = -\frac{\dfrac{\mu_e^{21}}{d_e} + \dfrac{\mu_i^{21}}{d_i}}{\alpha_{ei} < B^2 > \{\dfrac{\mu_e^{11}}{d_e} + \dfrac{\mu_i^{11}}{d_i}\} + 1}\,\frac{e\,\Gamma_{neo}}{V'(\psi)}. \tag{9.31}$$

This is the most general relation between bootstrap current and the neoclassical particle flux, it shows that the non-diagonal terms of the the viscosity matrix are the geometric factors which determine the magnitude of the bootstrap current. In general non-axisymmetric configurations these coefficients depend on the details of B on the magnetic surface, in principle, configurations can be constructed where these bootstrap coefficients μ^{21} are small or zero. It can be easily seen that these coefficients yield the correct limits of the collision-dominated plasma which have been found in chapter VI. In a collision-dominated plasma the operator L^{-1} is mainly the inverse of the collision operator; the velocity space integration and real space integration in the coefficients μ^{12}, μ^{21} decouple and we obtain

$$\mu^{12} \longrightarrow \tau < (\mathbf{B} \cdot \frac{\nabla B}{B})(\mathbf{V}_o \cdot \frac{\nabla B}{B}) > \tag{9.32}$$

Instead of writing the bootstrap current in terms of the particle fluxes the gradients E_e and E_i can be introduced by inverting Eq. (9.15) with respect to $\mathbf{\Lambda}$. After some simple algebra the result is

$$\mathbf{\Lambda} = [L_{22} + < B^2 > \alpha_{ei}\,\mathbf{S}]^{-1}\, L_{21}\,\mathbf{E} \tag{9.33}$$

and

$$\Lambda_e(\psi) - \Lambda_i(\psi) = -\frac{\mu_e^{21}\mu_i^{22}\, E_e - \mu_i^{21}\mu_e^{22}\, E_i}{< B^2 > \alpha_{ei}(\mu_e^{22} + \mu_i^{22}) + \mu_e^{22}\mu_i^{22}} \tag{9.34}$$

In this formulation the bootstrap current is proportional to the density gradients and the radial electric field which has to be determined from the condition of ambipolarity: $\Gamma_e = \Gamma_i$.

<u>The radial electric field</u>

In a steady state plasma the fluxes $\Gamma_e = \Gamma_i$ are equal to the particle source term, and therefore the density gradient and the radial electric field are determined by the fluxes. Eliminating the toroidal fluxes from the system (9.24) yields the following equation

$$\begin{aligned}\frac{e\,\Gamma_{neo}}{V'(\psi)}\,\mathbf{e}_- &= L_{11}\mathbf{E} - L_{12}\,[L_{22} + < B^2 > \alpha_{ei}\,\mathbf{S}]^{-1}\, L_{21}\,\mathbf{E} \\ &=: \; A\,\mathbf{E}\end{aligned} \tag{9.35}$$

where the matrix A is defined by this equation. Inverting this system leads to the gradients $\mathbf{E}$ as linear functions of the particle flux Γ_{neo}. The matrix $[L_{22} + < B^2 > \alpha_{ei}\,\mathbf{S}]^{-1}$ can be written in a simpler form

$$[L_{22} + < B^2 > \alpha_{ei}\,\mathbf{S}]^{-1} = \frac{1}{Det}\{Det(L_{22})\, L_{22}^{-1} + < B^2 > \alpha_{ei}\,\mathbf{I}\} \tag{9.36}$$

with

$$Det = < B^2 > \alpha_{ei}(\mu_i^{22} + \mu_e^{22}) + \mu_i^{22}\mu_e^{22} \quad ; \quad Det(L_{22}) = \mu_i^{22}\mu_e^{22} \tag{9.37}$$

and after some algebra the matrix A can be written as

$$\begin{aligned}A = \frac{1}{Det}\{&(Det(L_{22})\, L_{22}^{-1} + < B^2 > \alpha_{ei}\mathbf{1})\, D \\ &+ < B^2 > \alpha_{ei}(Det(L_{22}L_{22}^{-1}L_{11} - L_{12}\,\mathbf{N}\, L_{21})\}\end{aligned} \tag{9.38}$$

or in explicit form

$$A = \frac{1}{Det}\begin{pmatrix} (\mu_i^{22} + < B^2 > \alpha_{ei})d_e & 0 \\ 0 & (\mu_e^{22} + < B^2 > \alpha_{ei})d_i \end{pmatrix} + \frac{< B^2 > \alpha_{ei}}{Det}\begin{pmatrix} \mu_i^{22}\mu_e^{11} & -\mu_e^{21}\mu_i^{12} \\ -\mu_e^{21}\mu_i^{12} & \mu_i^{11}\mu_e^{22} \end{pmatrix} \tag{9.39}$$

This form of the matrix A is convenient for symmetric systems, since with $L_{22} = M^2\, L_{11}$ and $L_{12} = L_{21} = M\, L_{11}$ the matrix D is zero and A has the simple representation

$$A = \frac{M^2 < B^2 > \alpha_{ei}}{Det} Det(L_{11})\, \mathbf{S} \tag{9.40}$$

Equation (9.35) is used to calculate the radial electric field. As a function of the electric field and the density gradient $\mathbf{E}$ can be written

$$\mathbf{E} = \Phi'\, \mathbf{e}_+ + \frac{kT}{e}\frac{N'}{N}\mathbf{e}_- \tag{9.41}$$

By multiplying Eq. (9.35) with the vectors $\mathbf{e}_-$ and $\mathbf{e}_+$ two equations for the radial electric field and the density gradient can be derived. For this purpose Eq. (9.41) is inserted in (9.35) and the resulting equations are

$$\begin{aligned} 0 &= \Phi'\,(\mathbf{e}_+ \cdot A\mathbf{e}_+) + \frac{kT}{e}\frac{N'}{N}(\mathbf{e}_+ \cdot A\mathbf{e}_-) \\ 2\frac{e\,\Gamma_{neo}}{V'(\psi)} &= \Phi'\,(\mathbf{e}_- \cdot A\mathbf{e}_+) + \frac{kT}{e}\frac{N'}{N}(\mathbf{e}_- \cdot A\mathbf{e}_-) \end{aligned} \tag{9.42}$$

The first equation couples the electric field to the density gradient. Eliminating the radial field from this system leads to a linear relation between the particle flux and the density gradient. In explicit form the coefficients in these equations are

$$\begin{aligned} (\mathbf{e}_+ \cdot A\mathbf{e}_+) &= \frac{1}{Det}\{(\mu_i^{22} + < B^2 > \alpha_{ei})d_e + (\mu_e^{22} + < B^2 > \alpha_{ei})d_i \\ &\quad + < B^2 > \alpha_{ei}(\mu_i^{22}\mu_e^{11} + \mu_i^{11}\mu_e^{22} - \mu_e^{21}\mu_i^{12} - \mu_e^{12}\mu_i^{21})\} \end{aligned} \tag{9.43}$$

$$\begin{aligned} (\mathbf{e}_- \cdot A\mathbf{e}_-) &= \frac{1}{Det}\{(\mu_i^{22} + < B^2 > \alpha_{ei})d_e + (\mu_e^{22} + < B^2 > \alpha_{ei})d_i \\ &\quad + < B^2 > \alpha_{ei}(\mu_i^{22}\mu_e^{11} + \mu_i^{11}\mu_e^{22} + \mu_e^{21}\mu_i^{12} + \mu_e^{12}\mu_i^{21})\} \end{aligned} \tag{9.44}$$

$$\begin{aligned} (\mathbf{e}_+ \cdot A\mathbf{e}_-) &= \frac{1}{Det}\{(\mu_i^{22} + < B^2 > \alpha_{ei})d_e - (\mu_e^{22} + < B^2 > \alpha_{ei})d_i \\ &\quad + < B^2 > \alpha_{ei}(\mu_i^{22}\mu_e^{11} - \mu_i^{11}\mu_e^{22} + \mu_e^{21}\mu_i^{12} - \mu_e^{12}\mu_i^{21})\} \end{aligned} \tag{9.45}$$

$$\begin{aligned} (\mathbf{e}_- \cdot A\mathbf{e}_+) &= \frac{1}{Det}\{(\mu_i^{22} + < B^2 > \alpha_{ei})d_e - (\mu_e^{22} + < B^2 > \alpha_{ei})d_i \\ &\quad + < B^2 > \alpha_{ei}(\mu_i^{22}\mu_e^{11} - \mu_i^{11}\mu_e^{22} + \mu_e^{21}\mu_i^{12} - \mu_e^{12}\mu_i^{21})\} \end{aligned} \tag{9.46}$$

The radial electric field follows from Eq. (9.42)

$$\Phi'(\psi) = -\frac{(\mathbf{e}_+ \cdot A\mathbf{e}_-)}{(\mathbf{e}_+ \cdot A\mathbf{e}_+)} \frac{kT}{e} \frac{N'}{N} \tag{9.46}$$

and after replacing the E-field in the second equation the neoclassical particle flux is

$$-\frac{e\,\Gamma_{neo}}{V'(\psi)} = \frac{1}{2}\{(\mathbf{e}_- \cdot A\mathbf{e}_-) - \frac{(\mathbf{e}_+ \cdot A\mathbf{e}_-)^2}{(\mathbf{e}_+ \cdot A\mathbf{e}_+)}\} \frac{kT}{e} \frac{N'}{N} \tag{9.49}$$

In this form the particle flux is proportional to the density gradient.

The electric field is determined by the neoclassical particle fluxes since the Pfirsch-Schlüter fluxes are intrisically ambipolar. In symmetric systems, however, the electric field cannot be uniquely calculated because the matrix A is singular. As shown in Eq. (9.41) the matrix A degenerates to $A \propto \mathbf{S}$ and because of $\mathbf{S}\,\mathbf{e}_+ = 0$ and $\mathbf{e}_+ \cdot \mathbf{S}\mathbf{e}_- = 0$ the first equation of (9.42) is identically zero and the particle flux is given by

$$\frac{e\,\Gamma_{neo}}{V'(\psi)} = \frac{kT}{e} \frac{N'}{N} \frac{1}{2}(\mathbf{e}_- \cdot A\mathbf{e}_-) \tag{9.49}$$

where A is written as

$$A = \frac{< B^2 > \alpha_{ei}\mu_e^{11}\mu_i^{11}}{< B^2 > \alpha_{ei}(\mu_e^{11} + \mu_i^{11}) + M^2\mu_e^{11}\mu_i^{11}}\,\mathbf{S} \tag{9.50}$$

The radial electric field in symmetric systems is correlated to the integral toroidal fluxes, which follows from Eq. (9.23). For this reason only one of the two quantities, either $\Lambda_e(\psi) + \Lambda_i(\psi)$ or $\Phi'(\psi)$ can be chosen arbitrarily, the other one is determined by the parallel momentum balance.

Since the ion mass is much greater than the electron mass the relation holds $\mu_i^{11} >> \mu_e^{11}$ and therefore

$$(\mathbf{e}_- \cdot A\mathbf{e}_-) = 4\frac{< B^2 > \alpha_{ei}\mu_e^{11}}{< B^2 > \alpha_{ei} + M^2\mu_e^{11}}$$

It can be shown that the first term in the denominator is the leading one and therefore the neoclassical transport coefficient is reduced to

$$\frac{1}{2}(\mathbf{e}_- \cdot A\mathbf{e}_-) = 2\,\mu_e^{11} \tag{9.51}$$

The power dissipated by the lowest order flow is the sum of the frictional dissipation and the viscous dissipation and can be written as

$$P = < V_o^2 > \alpha_{ei}(\Lambda_e(\psi) - \Lambda_i(\psi))^2 - \sum_{e,i} < \mathbf{V}_j \cdot \nabla \cdot \pi_j >$$

or, after replacing the average viscous forces by Eqs. (9.13)

$$P = < V_o^2 > \alpha_{ei}(\Lambda_e(\psi) - \Lambda_i(\psi))^2 + \sum \mu_j^{11} E_j^2 + \sum \mu_j^{22} \Lambda_j(\psi)^2 + 2\sum \mu_j^{12} E_j \Lambda_j(\psi)$$

Minimisation of P with respect to $\Lambda_j(\psi)$ with $E_j(\psi)$ fixed yields the parallel momentum balance and thus defines the bootstrap current as the current minimizing the entropy production rate.

Plasma with external driving terms

In the preceeding analysis we have neglected external momentum sources ξ_j. In toroidal systems two different driving terms can exist; the first category is the inductive electric field in ohmically heated systems and the second one the momentum input by additional heating schemes. This encompasses HF-current drive and direct momentum input by neutral beam injection. With ohmic heating only the driving term $< \mathbf{E}_{ext} \cdot \mathbf{B} >$ is the same for electrons and ions and the source terms are

$$\begin{aligned} < \mathbf{B} \cdot \xi_e > &= eN < \mathbf{B} \cdot \mathbf{E}_{ext} > \\ < \mathbf{B} \cdot \xi_i > &= -eN < \mathbf{B} \cdot \mathbf{E}_{ext} > \end{aligned} \tag{9.52}$$

In steady state the scalar potential of the inductive electric field is single valued in the poloidal coordinate and therefore $< \mathbf{V}_o \cdot \mathbf{E}_{ext} >= 0$. The momentum balance equations with external loop voltage are written as

$$\begin{aligned} -eN < \mathbf{B} \cdot \mathbf{E}_{ext} > \mathbf{e}_- &= L_{22} \mathbf{\Lambda} + L_{21}\mathbf{E} + < B^2 > \alpha_{ei} \mathbf{S\Lambda} \\ \frac{e\,\Gamma_{neo}}{V'(\psi)} \mathbf{e}_- &= L_{12}\mathbf{\Lambda} + L_{11}\mathbf{E} \end{aligned} \tag{9.52}$$

Elimination of $\mathbf{E}$ yields an equation for the toroidal fluxes

$$-eN < \mathbf{B} \cdot \mathbf{E}_{ext} > L_{11}\mathbf{e}_- - \frac{e\,\Gamma_{neo}}{V'(\psi)} L_{21}\mathbf{e}_- = < B^2 > \alpha_{ei}\, L_{11}\mathbf{S\Lambda} + D\mathbf{\Lambda} \tag{9.53}$$

from which the toroidal current $eN(\Lambda_e(\psi) - \Lambda_i(\psi))$ can be calculated as a sum of the inductively driven current and the neoclassical bootstrap current.

After eliminating the toroidal fluxes the total radial particle flux is calculated from

$$\frac{e\,\Gamma}{V'(\psi)} \mathbf{e}_- = < V_o^2 > \alpha_{ei}\, \mathbf{SE} + A\mathbf{E} + L_{12}\frac{Det(L_{22})}{Det} L_{22}^{-1}\, eN < \mathbf{B} \cdot \mathbf{E}_{ext} > \mathbf{e}_- \tag{9.54}$$

which after writing the vector $\mathbf{E}$ in the form Eq. (9.41) and replacing Γ by Γ_{neo} is transferred to

$$\begin{aligned} \Phi'(\mathbf{e}_+ \cdot A\mathbf{e}_+) + \frac{kT}{e}\frac{N'}{N}(\mathbf{e}_+ \cdot A\mathbf{e}_-) &= -\mathbf{e}_+ \cdot L_{12}\frac{Det(L_{22})}{Det} L_{22}^{-1}\, eN < \mathbf{B} \cdot \mathbf{E}_{ext} > \mathbf{e}_- \\ \Phi'(\mathbf{e}_- \cdot A\mathbf{e}_+) + \frac{kT}{e}\frac{N'}{N}(\mathbf{e}_- \cdot A\mathbf{e}_-) &= -\mathbf{e}_- \cdot L_{12}\frac{Det(L_{22})}{Det} L_{22}^{-1}\mathbf{e}_-\, eN < \mathbf{B} \cdot \mathbf{E}_{ext} > \\ &\quad + 2\frac{e\,\Gamma_{neo}}{V'(\psi)} \end{aligned} \tag{9.55}$$

In symmetric configurations the inhomogeneous term on the left hand side of the first equation is zero since L_{12} and L_{22} are proportional and $\mathbf{e}_+ \cdot \mathbf{e}_- = 0$. Therefore all terms in the first equations vanish identically and the radial electric field remains undetermined.

Additional heating in stellarators and tokamaks provides another source of momentum input ξ_j. The energy and momentum input during heating leads to a distortion of the distribution function and can enhance or reduce the anisotropy of the distribution function. Therfore the difference $p_{\parallel} - p_{\perp}$ can be decomposed in two parts, the first one describing the anisotropy caused by the particle drifts and the second one describing the modification of the distribution function by fast particles. Fast particles lead to an additional term π_f in the pressure tensor and the vector ξ_j consists of two terms, $\xi_o - \nabla \cdot \pi_f$, where ξ_o is the first order moment of the source term in the Fokker- Planck equation. Different heating schemes have different influence on the source term ξ_j. Parallel neutral beam injection will mainly give rise to $< \mathbf{B} \cdot \xi_j >$ and with perpendicular injection the term $< \mathbf{V}_o \cdot \xi >$ dominates. With perpendicular neutral beam injection fast particles can be trapped in local magnetic mirrors leading to a fast ion population in this loss cone region. The perpendicular pressure anisotropy of these fast particles can also be described by π_f and $\xi = -\nabla \cdot \pi_f$. Because of this effect of perpendicular injection on the perpendicular momentum balance the radial electric field can be controlled by neutral beam heating. As a function of the specific heating scheme, the momentum source terms ξ_j can be different for electrons and ions and in general neither parallel nor poloidal momentum is conserved by these sources.

$$\sum_j < \mathbf{B} \cdot \xi_j > \neq 0 \quad ; \quad \sum_j < \mathbf{V}_o \cdot \xi_j > \neq 0$$

In contrast to ohmic heating these external driving terms can lead to strong toroidal or poloidal rotation if the slowing down mechanism is small. To simplify the algebra the external source terms are combined to

$$\begin{aligned} \mathbf{X} &=: \{< \mathbf{B} \cdot \xi_e >, < \mathbf{B} \cdot \xi_i >\} \\ \mathbf{Y} &=: \{< \mathbf{V}_o \cdot \xi_e >, < \mathbf{V}_o \cdot \xi_i >\} \end{aligned} \tag{9.54}$$

and the balance equations are

$$\begin{aligned} -\mathbf{X} &= L_{22}\,\mathbf{\Lambda} + L_{21}\mathbf{E} + < B^2 > \alpha_{ei}\,\mathbf{S\Lambda} \\ \frac{e\,\Gamma_{neo}}{V'(\psi)}\,\mathbf{e}_- - \mathbf{Y} &= L_{12}\mathbf{\Lambda} + L_{11}\mathbf{E} \end{aligned} \tag{9.55}$$

and after eliminating the gradient $\mathbf{E}$ the toroidal fluxes are to be calculated from

$$-L_{11}\mathbf{X} - L_{21}\left(\frac{e\,\Gamma_{neo}}{V'(\psi)}\,\mathbf{e}_- - \mathbf{Y}\right) = < B^2 > \alpha_{ei}\,L_{11}\mathbf{S\Lambda} + D\mathbf{\Lambda} \tag{9.56}$$

This generalizes equation 9.25. If the matrix D is non-singular the toroidal fluxes and thus the bootstrap current can be uniquely determined from this equation. In case of symmetric configurations, however, $D = 0$, and the toroidal fluxes become arbitrarily large if the condition

$$\mathbf{e}_+ \cdot \left\{L_{11}\mathbf{X} + L_{21}\left(\frac{e\,\Gamma_{neo}}{V'(\psi)}\,\mathbf{e}_- - \mathbf{Y}\right) = 0\right.$$

is not satisfied. In symmetric systems this condition can be reduced to

$$\mathbf{e}_+ \cdot \left(\mathbf{X} - M\mathbf{Y}\right) = 0$$

As shown above, this condition is satisfied in case of ohmic heating; in case of tangential neutral beam heating the momentum is mainly delivered to the ions and the plasma begins to rotate toroidally. A similar singularity can arise in the radial electric field or the poloidal rotation of the plasma. After eliminating the toroidal fluxes from Eqs. (9.55) the resulting equation relating particle fluxes to the gradients $\mathbf{E}$ is

$$\frac{e\,\Gamma}{V'(\psi)}\,\mathbf{e}_- + \mathbf{Y} - L_{12}[L_{22} + < B^2 > \alpha_{ei}\,\mathbf{S}]^{-1}\mathbf{X} = < V_o^2 > \alpha_{ei}\,\mathbf{SE} + A\mathbf{E} \tag{9.57}$$

The momentum source terms modify the particle balance equation and generate an extra particle flux through the magnetic surface. As a consequence the ambipolar condition is modified which is seen from the following equations

$$\begin{aligned} \mathbf{e}_+ \cdot \mathbf{Z} &= \Phi'\,(\mathbf{e}_+ \cdot A\mathbf{e}_+) + \frac{kT}{e}\frac{N'}{N}\,(\mathbf{e}_+ \cdot A\mathbf{e}_-) \\ 2\frac{e\,\Gamma_{neo}}{V'(\psi)} + \mathbf{e}_- \cdot \mathbf{Z} &= \Phi'\,(\mathbf{e}_- \cdot A\mathbf{e}_+) + \frac{kT}{e}\frac{N'}{N}\,(\mathbf{e}_- \cdot A\mathbf{e}_-) \end{aligned} \tag{9.58}$$

Here, the abbreviation has been used

$$\mathbf{Z} =: L_{12}[L_{22} + < B^2 > \alpha_{ei}\,\mathbf{S}]^{-1}\mathbf{X} - \mathbf{Y}$$

This vector describes the poloidal momentum sources provided by the additional heating mechanisms. After eliminating the radial electric field from this equation the relation between particle flux and density gradient is

$$\begin{aligned} 2\frac{e\,\Gamma_{neo}}{V'(\psi)} &= \frac{[(\mathbf{e}_- \cdot A\mathbf{e}_-)(\mathbf{e}_+ \cdot A\mathbf{e}_+) - (\mathbf{e}_+ \cdot A\mathbf{e}_-)(\mathbf{e}_- \cdot A\mathbf{e}_+)]}{(\mathbf{e}_+ \cdot A\mathbf{e}_+)}\,\frac{kT}{e}\,\frac{N'}{N} \\ &+ \frac{(\mathbf{e}_+ \cdot A\mathbf{e}_-)}{(\mathbf{e}_+ \cdot A\mathbf{e}_+)}\,(\mathbf{e}_+ \cdot \mathbf{Z}) - \mathbf{e}_- \cdot \mathbf{Z} \end{aligned} \tag{9.59}$$

and the corresponding equation for the electric field

$$\begin{aligned} \Phi'(\psi)\,[(\mathbf{e}_+ \cdot A\mathbf{e}_+)(\mathbf{e}_- \cdot A\mathbf{e}_-) - (\mathbf{e}_+ \cdot A\mathbf{e}_-)(\mathbf{e}_- \cdot A\mathbf{e}_+)] &= -2\frac{e\,\Gamma_{neo}}{V'(\psi)}(\mathbf{e}_+ \cdot A\mathbf{e}_-) \\ &(\mathbf{e}_+ \cdot \mathbf{Z})(\mathbf{e}_- \cdot A\mathbf{e}_-) - (\mathbf{e}_- \cdot \mathbf{Z})(\mathbf{e}_+ \cdot A\mathbf{e}_-) \end{aligned} \tag{9.60}$$

The radial electric field and the radial particle flux are finite if the matrix A is non-singular. As has been shown above this is not the case in symmetric systems. There, the coefficients $(\mathbf{e}_+ \cdot A\mathbf{e}_+)$, $(\mathbf{e}_- \cdot A\mathbf{e}_+)$ and $(\mathbf{e}_+ \cdot A\mathbf{e}_-)$ are zero and $(\mathbf{e}_- \cdot A\mathbf{e}_-) \neq 0$. The electric field becomes arbitrarily large if $\mathbf{e}_+ \cdot \mathbf{Z} \neq 0$. The particle flux consists of a term driven by the density gradient and two further terms caused by the momentum sources. The term proportional to $(\mathbf{e}_+ \cdot \mathbf{Z})$ is undetermined in symmetric systems and can be arbitrarily large close to symmetry. Here, a fundamental difference between ohmic heating and non-ohmic heating exists. As has been shown above, with ohmic heating only the term $\mathbf{e}_+ \cdot \mathbf{Z}$ is zero in symmetric systems and the electric field can be finite. In non-ohmic heating schemes, however, there can be a net toroidal momentum input and toroidal rotation or the radial electric field diverge. A small damping mechanism caused by the magnetic field ripple may prevent the infinite solutions, the effect on the radial losses as described by Eq. (9.60) can still be large.

Tokamak

As a special case of a symmetric configuration we shall consider the axisymmetric tokamak in more detail. The matrices L_{ik} are related to each other in the following way

$$L_{22} = \left(\frac{\iota}{V'(\psi)}\right)^2 L_{11} \quad ; \quad L_{12} = L_{21} = -\frac{\iota}{V'(\psi)} L_{11} \tag{9.61}$$

Instead of the vectors $\mathbf{E}$ and $\mathbf{\Lambda}$ the following poloidal and toroidal fluxes (see Eq. 1.19) are more appopriate in tokamaks

$$\mathbf{u}_p = \iota\mathbf{\Lambda} - \mathbf{E}\,V'(\psi) \tag{9.62}$$

The balance equations 9.55 are written in this case

$$\begin{aligned} -\mathbf{X} &= \frac{\iota}{(V'(\psi))^2} L_{11}\,\mathbf{u}_p + < B^2 > \alpha_{ei}\mathbf{S}\mathbf{u}_t \\ \frac{e\Gamma_{neo}}{V'(\psi)}\mathbf{e}_- - \mathbf{Y} &= -\frac{1}{V'(\psi)} L_{11}\,\mathbf{u}_p \end{aligned} \tag{9.63}$$

This system couples the poloidal and toroidal fluxes to the external momentum sources and the neoclassical particle flux. If ohmic heating is the only momentum source ($\mathbf{Y} = 0$ and $\mathbf{X} \propto \mathbf{e}_-$) the poloidal fluxes are calculated from

$$\begin{aligned} -\frac{e\Gamma_{neo}}{V'(\psi)} &= \frac{1}{V'(\psi)}(\mu_e^{11}\,u_{p,e} - \mu_i^{11}\,u_{p,i}) \\ 0 &= \frac{1}{V'(\psi)}(\mu_e^{11}\,u_{p,e} + \mu_i^{11}\,u_{p,i}) \end{aligned} \tag{9.64}$$

The result is

$$\mu_e^{11}\,u_{p,e} = -e\Gamma_{neo} \quad ; \quad \mu_i^{11}\,u_{p,i} = e\Gamma_{neo} \tag{9.65}$$

The poloidal fluxes are uniquely determined by the neoclassical particle flux Γ_{neo} in a tokamak, whereas the toroidal fluxes are not uniquely determined. Replacing $\mathbf{u}_p$ in the first equation of (9.63) yields a singular system for $\mathbf{u}_t$, the reason being the singularity of the matrix $\mathbf{S}$. The difference $u_{t,e} - u_{t,i}$ can be correlated uniquely to the neoclassical particle flux, but the sum $u_{t,e} + u_{t,i}$ remains arbitrary which implies that also the radial electric field is arbitrary. This ambiguity can be removed by the small ripple effect of the TF-coils which introduces viscous damping in toroidal motion. In this case $L_{12} + t/V'(\psi)L_{11} \neq 0$ and $L_{22} + t/V'(\psi)L_{21} \neq 0$ and the system (9.55) is written as

$$
\begin{aligned}
-\mathbf{X} &= (L_{22} + L_{21}\frac{t}{V'(\psi)})\,\mathbf{u}_t - \frac{1}{V'(\psi)}\,L_{21}\,\mathbf{u}_p + < B^2 > \alpha_{ei}\mathbf{S}\mathbf{u}_t \\
\frac{e\,\Gamma_{neo}}{V'(\psi)}\,\mathbf{e}_- - \mathbf{Y} &= (L_{12} + L_{11}\frac{t}{V'(\psi)})\mathbf{u}_t - \frac{1}{V'(\psi)}\,L_{11}\,\mathbf{u}_p
\end{aligned}
\tag{9.66}
$$

The small ripple effect couples the toroidal flow $\mathbf{u}_t$ to the poloidal momentum balance. The first equation can be solved for the toroidal flux $\mathbf{u}_t$ in terms of the toroidal momentum source $\mathbf{X}$ and the poloidal flux $\mathbf{u}_p$. With this result the toroidal flux $\mathbf{u}_t$ is eliminated in the second equation and the result is

$$
\frac{e\Gamma_{neo}}{V'(\psi)}\,\mathbf{e}_- - \mathbf{Y} = -\mathbf{R}\mathbf{X} + \frac{1}{V'(\psi)}(\mathbf{R}\,L_{21} - L_{11})\,\mathbf{u}_p
$$

where the matrix $\mathbf{R}$ is defined by

$$
\mathbf{R} = (L_{12} + L_{11}\frac{t}{V'(\psi)})\big(L_{22} + L_{21}\frac{t}{V'(\psi)} + < B^2 > \alpha_{ei}\big)^{-1}.
$$

This equation shows how external momentum sources $\mathbf{X}$ and $\mathbf{Y}$ modify the particle loss in tokamaks. The matrix $\mathbf{R}$ is the quotient of two nearly singular matrices, and therefore $\mathbf{R}$ depends critically on the toroidal ripple viscosity. Eliminating the poloidal fluxes $\mathbf{u}_p$ yields

$$
-L_{11}\mathbf{X} - L_{21}\big(\frac{e\Gamma_{neo}}{V'(\psi)}\mathbf{e}_- - \mathbf{Y}\big) = < B^2 > \alpha_{ei}L_{11}\mathbf{S}\mathbf{u}_t + D\mathbf{u}_t \tag{9.67}
$$

which is Eq. (9.56) in special form for tokamaks ($D = L_{22}L_{11} - L_{21}L_{12}$). The matrix D is non-singular because of the small ripple effect. As a consequence, the toroidal fluxes are uniquely determined by the external momentum sources, however the fluxes can be rather large since the damping mechanism is small. The Ware pinch mechanism couples the toroidal fluxes to the radial fluxes and therefore external momentum sources can strongly affect the radial particle loss.

X. Transport coefficients

In order to evaluate the transport coefficients μ^{ik} the Hamada coordinate system is the appropriate one. By introducing the length L_o of the magnetic axis as the length scale and the thermal velocity v_{th} as a reference velocity the operator L is written in the form

$$L = \frac{v_{th}}{L_o}\{u_{\|}(\iota\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\varsigma}) + V_E\frac{\partial}{\partial\eta} - \nu^\star C_T\} \tag{10.1}$$

with

$$\mathbf{u} = \frac{\mathbf{v}}{v_{th}} \quad ; \quad V'(\psi) \approx \frac{L_o}{B_o} \quad ; \quad \nu^\star = \frac{\nu L_o}{v_{th}}$$

C_T is the dimensionless collision operator. Several approximations have been made to obtain this formulation

$$\frac{1}{V'(\psi)B} \approx \frac{1}{L_o} \quad ; \quad V_E = L_o\frac{\mathbf{V}_E\cdot\nabla\eta}{v_{th}} =: V_E(\psi)$$

The operator $L_o/v_{th}\mathbf{v}_E\cdot\nabla$ has been approximated by $V_E\frac{\partial}{\partial\eta}$. $\nu^\star$ is the collisionality. The Maxwellian is normalized to the density N. Redefining the terms ω_1, ω_2 by

$$\begin{aligned}\omega_1 &= (u_{\|}^2 - \frac{1}{2}u_\perp^2)\,\frac{\partial}{\partial\eta}ln\,B\\ \omega_2 &= (u_{\|}^2 - \frac{1}{2}u_\perp^2)\,(\iota\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\varsigma})\,ln\,B\end{aligned} \tag{10.2}$$

yields the transport coefficients

$$\begin{aligned}-\mu^{11} &= A_o < \int \omega_1 L^{-1} f_o\omega_1\, d^3u >\\ -\mu^{22} &= A_o(\frac{1}{V'(\psi)})^2 < \int \omega_2 L^{-1} f_o\omega_2\, d^3u >\\ -\mu^{21} &= A_o\frac{1}{V'(\psi)} < \int \omega_2 L^{-1} f_o\omega_1\, d^3u >\end{aligned} \tag{10.3}$$

with

$$A_o = \frac{m_j^2}{kT}v_{th}^3 L_o \propto \sqrt{m_j\,kT}\,L_o$$

Here L is given in dimensionless form

$$L = u_{\|}(\iota\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\varsigma}) + V_E\frac{\partial}{\partial\eta} - \nu^\star C_T \tag{10.4}$$

The plateau limit

To evaluate the transport coefficients in the plateau limit is standard technique since the approximation $C_T \approx 1$ can be made and $u_{\|}$ is the independent variable in velocity space. The kinetic equation $Lf_1 = f_o\omega_1$ can be solved by Fourier transform in the Hamada coordinates η and ς . Let $ln\, B$ be given in a series

$$ln\, B = \sum_{l,m=0}^{\infty} a_{l,m} \cos(l\eta - m\varsigma) + b_{l,m} \cos(l\eta + m\varsigma) \tag{10.5}$$

From this general form of $ln\, B$ the coefficients μ^{ik} can be computed

$$\mu^{11} = A_o \sum_0^{\infty} \{ g^{(-)}_{l,m}(\nu^\star) l^2 a^2_{l,m} + g^{(+)}_{l,m}(\nu^\star)\, l^2 b^2_{l,m} \} \tag{10.6}$$

with

$$g^{(-)}_{l,m}(\nu^\star) = \frac{1}{4}\int_{\infty}^{\infty} \frac{\nu^\star (u_{\|}^2 - \frac{1}{2}u_{\perp}^2)^2 f_o(u_{\|}^2)}{(\nu^\star)^2 + [u_{\|}(l\iota - m) + V_E l]^2} du_{\|}$$

$$g^{(+)}_{l,m}(\nu^\star) = \frac{1}{4}\int_{\infty}^{\infty} \frac{\nu^\star (u_{\|}^2 - \frac{1}{2}u_{\perp}^2)^2 f_o(u_{\|}^2)}{(\nu^\star)^2 + [u_{\|}(l\iota + m) + V_E l]^2} du_{\|} \tag{10.7}$$

If the electric field in the denominator is neglected, a singularity arises in the limit $\nu^\star \longrightarrow 0$.

$$V_E = 0; \nu^\star \longrightarrow 0 \succ\; g^{(-)}_{l,m}(\nu^\star) \longrightarrow \frac{1}{|l\iota - m|}$$

$$V_E = 0; \nu^\star \longrightarrow 0 \succ\; g^{(+)}_{l,m}(\nu^\star) \longrightarrow \frac{1}{|l\iota + m|} \tag{10.8}$$

and

$$\mu^{11} \longrightarrow A_o \sum \{ \frac{l^2}{|l\iota - m|} a^2_{l,m} + \frac{l^2}{|l\iota + m|} b^2_{l,m} \} \tag{10.9}$$

These coefficients diverge on rational magnetic surfaces. These singularities are removed with finite electric drift, since in that case with $\nu^\star \longrightarrow 0$ the resonance occurs at

$$u_{\|,res} = -\frac{V_E\, l}{l\iota - m}.$$

On rational magnetic surfaces $u_{\|,res} \longrightarrow \infty$ and because of the exponential decay in the Maxwellian $f_o(u_{\|}^2)$ the integral $g^{(-)}_{l,m}(\nu^\star)$ is zero on rational magnetic surfaces. A similar result holds for μ^{22}.

$$\mu^{22} = A_o(\frac{1}{V'(\psi)})^2 \sum_0^{\infty} \{ g^{(-)}_{l,m}(\nu^\star)(l\iota - m)^2\, a^2_{l,m} + g^{(+)}_{l,m}(\nu^\star)\,(l\iota + m)^2\, b^2_{l,m} \} \tag{10.10}$$

and

$$\mu^{21} = A_o \frac{1}{V'(\psi)} \sum_0^{\infty} \{g^{(-)}_{l,m}(\nu^*) l(\iota - m)\, a^2_{l,m} + g^{(+)}_{l,m}(\nu^*)\, l(\iota + m)\, b^2_{l,m}\} \tag{10.11}$$

With $V_E = 0$ and $\nu^* \longrightarrow 0$ the result is

$$\mu^{21} = A_o \frac{1}{V'(\psi)} \sum \{ l\, a^2_{l,m}\, sign\,(\iota - m) + l\, b^2_{l,m}\, sign\,(\iota + m)\} \qquad (10.12).$$

With zero electric drift the bootstrap factor μ^{21} exhibits a discontinuity which is removed with finite electric field. The function $g^{(-)}_{l,m}(\nu^*), \nu^* \longrightarrow 0$ is finite with finite V_E and therefore the sum in Eq. (10.12) does not comprise discontinuous terms. In those terms with $\iota - m \neq 0$ the electric field can be neglected in Eq. (10.11) and Eq. (10.12) can be used as an approximation of μ^{21} if in the sum the resonant terms are omitted. In this approximation the transport coefficients μ are nearly independent of the electric field and are completely determined by mod B on the magnetic surfaces.

In a paper of Rodriguez-Solano and Shaing[1] a similar equation for the bootstrap current is given. The geometrical factor derived in that paper is similar to Eq. (10.12), however an additional term exists which is of the order $\iota r^2/R^2$ and therefore negligible. In Ref.[20][2] the bootstrap current is characterized by two geometrical factors μ^p and μ^t, μ^p is the same as Eq. (10.12) in the present paper.[3]

Since the coefficient μ^{21} is the sum of positive and negative terms, there is a chance to find configurations with zero or small bootstrap current if the Fourier spectrum of $mod\, B$ is appropriately chosen. As already pointed out in the paper of Callen et al. Ref.[7] , the helical harmonics in $mod\, B$ can compensate the effect of the toroidal curvature which is described by the coefficients $a_{l,0}$ and $b_{l,0}$. Since the helical harmonics are a function of the radial coordinate, the coefficient μ^{12} vanishes on a particular magnetic surface and remains finite elsewhere. Examples of stellarator configurations with zero or small bootstrap current on all magnetic surfaces are given in Ref.[20] . In a helical axis stellarator of the Helias type the effect of toroidal curvature in Eq. (10.12) is compensated on nearly all magnetic surfaces. In Fig. 2 a magnetic surface of such a configuration is shown together with the $mod\, B$-contours on this surface.

In the long-mean-free-path regime the solution g of the kinetic equation has a stronger dependence on the radial electric field than in the plateau regime. This is mainly because the drift of the trapped particles is affected by the poloidal $\mathbf{E}\times\mathbf{B}$- drift. The radial

[1] E. Rodriguez-Solano Ribeiro, K.C. Shaing

[2] K.C. Shaing, S.P. Hirshman and J.D. Callen, *Phys. Fluids* **29** (1986), 521

[3] In Ref.[20] the parallel viscous forces are linear in the poloidal and toroidal particle fluxes u_p and u_t, the μ^p and μ^t as coefficients. The authors relate these fluxes linearly to the density and temperature gradients (see Eqs. 15, 16 and 22 in Ref.[20]) and consequently all neoclassical fluxes (Eqs. 30 -35) are linear in μ^p and μ^t. The toroidal fluxes, however, are carrying the bootstrap current and therefore two coupled equations for u^t_e and u^t_i have to be solved and the bootstrap current has to be calculated from the difference $u^t_e - u^t_i$. Therefore, only μ^p is the relevant geometrical factor.

transport coefficients are computed from the bounce-averaged kinetic equation, whereas - as Shaing and Callen have shown - only the non-bounce averaged part of the distribution function is relevant for the bootstrap current. This can easily shown in the coefficient μ^{21} by partial integration. For the non- bounce averaged part of the distribution function δg the kinetic equation $Lg = h$ yields

$$[u_{\|}(\iota\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\varsigma}) + V_E\frac{\partial}{\partial\eta}] - \nu^* C(\delta g) - \nu^*[C(\bar{g}) + \overline{C(g)}] = h - \bar{h}$$

where

$$\bar{g} = \frac{\oint \frac{dl}{u_{\|}}}{\oint \frac{dl}{u_{\|}}}$$

is the bounce-averaging process between turning points of trapped particles. It can be seen that the electric drift occurs in the kinetic equation for δg in the same manner as in the bounce- averaged equation. The electric is mainly effective at the turning points of the trapped particles where $u_{\|} = 0$. Therefore it has to be expected that in general stellarator configurations the radial electric field also modifies the bootstrap current in the long-mean-free-path regime.

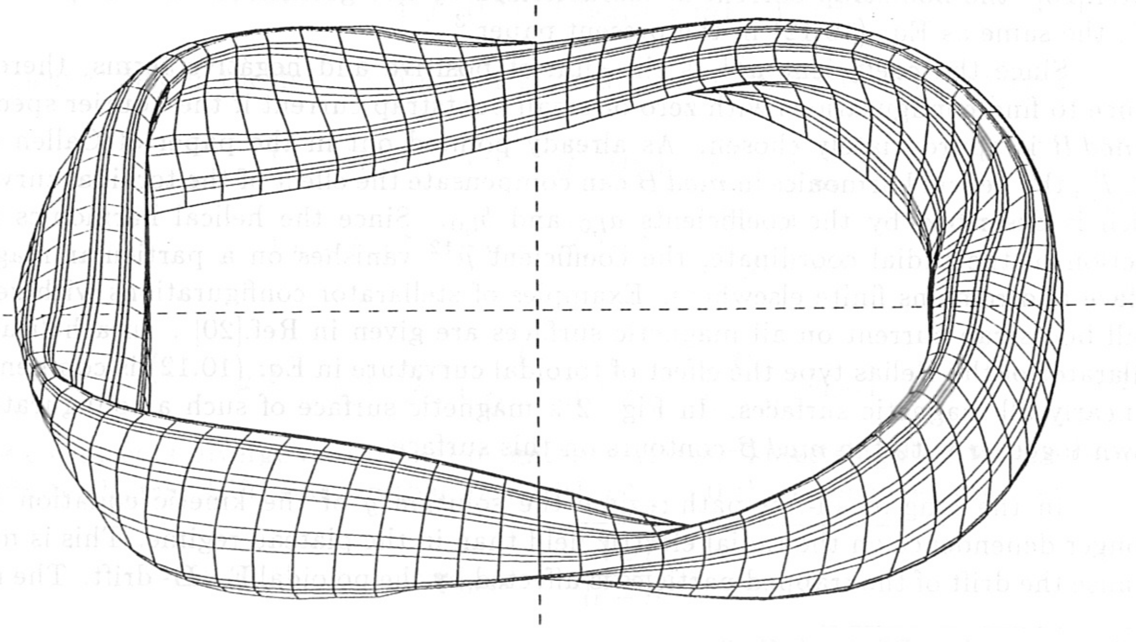

Fig .2: Magnetic surface of a 4-period Helias configuration. The bootstrap current in the plateau regime is negligibly small.

Non-isothermal plasma

In non-isothermal plasmas the energy balance equations have to be taken into account, as has been already studied in chapter **VII** for the case of symmetric systems. The flux friction relations are given in Eqs. (7.15). In general non-symmetric configurations the viscous forces have to be calculated from the kinetic equation. The kinetic equation with thermal fluxes $\mathbf{q}^o$ is

$$L\,g_1 \;=\; (v_\parallel^2 - \frac{1}{2}v_\perp^2)\, f_o\,\{(\mathbf{V}_j\cdot\frac{\nabla B}{B}) + (\frac{2}{5}\frac{v^2}{v_{th}^2} - 1)\mathbf{q}_j^o\cdot\frac{\nabla B}{B}\} \tag{10.13}$$

The lowest order thermal flux in the magnetic surface is

$$\mathbf{q}_j^o \;=\; \frac{5}{2}\frac{P_j}{e_j}\,T_j'(\psi)\,\mathbf{V}_o + X_j\mathbf{B} \tag{10.14}$$

X_j is defined by equation (7.18). In the following the shorter notation

$$K_j \;=\; \frac{5}{2}\frac{P_j}{e_j}\,T_j'(\psi)$$

and the definitions

$$\begin{aligned} w_1 &= (v_\parallel^2 - \frac{1}{2}v_\perp^2)(\mathbf{V}_o\cdot\frac{\nabla B}{B}) \quad ; \quad w_3 = w_1(\frac{2}{5}\frac{v^2}{v_{th}^2} - 1) \\ w_2 &= (v_\parallel^2 - \frac{1}{2}v_\perp^2)(\mathbf{B}\cdot\frac{\nabla B}{B}) \quad ; \quad w_4 = w_2(\frac{2}{5}\frac{v^2}{v_{th}^2} - 1) \end{aligned} \tag{10.15}$$

and, after defining the transport coefficients μ^{ik} in Eq. (9.14) with $i,k = 1,..4$ the parallel and poloidal viscous forces (see Eq. (9.13)) are

$$-<\mathbf{B}\cdot\nabla\cdot\pi_j> = \mu_j^{21}\,E_j(\psi) + \mu_j^{22}\,\Lambda_j(\psi) + \mu_j^{23}\,K_j(\psi) + \mu_j^{24}\,X_j \tag{10.16}$$

and

$$-<\mathbf{V}_o\cdot\nabla\cdot\pi_j> = \mu_j^{11}\,E_j(\psi) + \mu_j^{12}\,\Lambda_j(\psi) + \mu_j^{13}\,K_j(\psi) + \mu_j^{14}\,X_j \tag{10.17}$$

The energy weighted anisotropic pressure is

$$\Theta_\parallel - \Theta_\perp \;=\; \frac{5}{2}\int g_1\,(v_\parallel^2 - \frac{1}{2}v_\perp^2)(\frac{2}{5}\frac{v^2}{v_{th}^2} - 1)\,d^3v \tag{10.18}$$

and the tangential heat viscous forces are

$$-<\mathbf{B}\cdot\nabla\cdot\Theta_j> = \mu_j^{41}\,E_j(\psi) + \mu_j^{42}\,\Lambda_j(\psi) + \mu_j^{43}\,K_j(\psi) + \mu_j^{44}\,X_j \tag{10.19}$$

and

$$-<\mathbf{V}_o\cdot\nabla\cdot\Theta_j> = \mu_j^{31}\,E_j(\psi) + \mu_j^{32}\,\Lambda_j(\psi) + \mu_j^{33}\,K_j(\psi) + \mu_j^{34}\,X_j \tag{10.20}$$

The coefficients μ_j^{ik} define the diagonal matrices $L_{ik}, i, k = 1..4$ (see Eq. (9.19) and (9.20)).

XI. Neoclassical plasma with temperature gradients

The transport coefficients derived in the preceeding section will be used to establish the flux friction relations in a multi-species plasma. The tangential forces on the magnetic surface, given in Eqs. (10.16) - (10.20), have to be balanced by the friction forces. These are given in Eqs. (7.11) and (7.12) and, using matrix notation, the general flux- friction relations for the parallel momentum balance are

$$< B^2 > \{l_{11}\mathbf{\Lambda} - \frac{2}{5}l_{12}\mathbf{X}\} + L_{21}\mathbf{E} + L_{22}\mathbf{\Lambda} + L_{23}\mathbf{K} + L_{24}\mathbf{X} = 0 \tag{11.1}$$

and

$$- < B^2 > \{l_{21}\mathbf{\Lambda} - \frac{2}{5}l_{22}\mathbf{X}\} + L_{41}\mathbf{E} + L_{42}\mathbf{\Lambda} + L_{43}\mathbf{K} + L_{44}\mathbf{X} = 0 \tag{11.2}$$

The vectors $\mathbf{E}, \mathbf{K}, \mathbf{\Lambda}$ and $\mathbf{X}$ are defined by $\{E_j\}, \{K_j\}, \{\Lambda\}$ and $\{X_j\}$. These equations have to be supplemented by the particle and energy balance equations (7.25) and (7.26)

$$\frac{1}{V'(\psi)}\mathbf{\Gamma}_{neo} = L_{11}\mathbf{E} + L_{12}\mathbf{\Lambda} + L_{13}\mathbf{K} + L_{14}\mathbf{X} \tag{11.3}$$

$$\frac{1}{V'(\psi)}\mathbf{q}_{neo} = L_{31}\mathbf{E} + L_{32}\mathbf{\Lambda} + L_{33}\mathbf{K} + L_{34}\mathbf{X} \tag{11.4}$$

The vector $\mathbf{\Gamma}_{neo}$ has the components $e_j\Gamma_j - V'(\psi) < \mathbf{V}_o \cdot \mathbf{R}_j >$. $\mathbf{q}_{neo}$ is defined in the same way.

These equations generalize equations (9.15) and (9.16) to a plasma with temperature gradients. The symmetric case can easily be recovered by observing that $L_{2k} = M\, L_{1k}$ and $L_{4k} = M\, L_{3k}$ which leads to

$$\begin{aligned} < B^2 > \{l_{11}\,\mathbf{\Lambda} - \frac{2}{5}l_{12}\mathbf{X}\} + \frac{M}{V'(\psi)}\mathbf{\Gamma}_{neo} &= 0 \\ - < B^2 > \{l_{21}\,\mathbf{\Lambda} - \frac{2}{5}l_{22}\,\mathbf{X}\} + \frac{M}{V'(\psi)}\mathbf{q}_{neo} &= 0 \end{aligned} \tag{11.5}$$

This system is equivalent to Eqs. (7.15) with the parallel viscosities being replaced by the perpendicular viscosities.

The system (11.1)-(11.4) is valid for an arbitrary number of particle species, it can be solved for the toroidal fluxes in terms of the forces $\mathbf{E}$ and $\mathbf{K}$. For this purpose Eq. (11.2) is inverted with respect $\mathbf{X}$, which is always possible since the matrices L_{44} and l_{22} are non-singular. Replacing the parallel thermal fluxes in Eq. (11.1) by this solution yields a linear relation between $\mathbf{\Lambda}, \mathbf{E}$ and $\mathbf{K}$ which then can be inverted with respect to $\mathbf{\Lambda}$.

Instead of the forces we can introduce the fluxes Γ_{neo} and q_{neo} as independent variables by inverting the equations 11.3 and 11.4 with respect to $\mathbf{E}$ and $\mathbf{K}$. After elimination of the parallel fluxes $\mathbf{X}$ the toroidal fluxes $\mathbf{\Lambda}$ are linear functions of $\mathbf{\Gamma}_{neo}$ and $\mathbf{q}_{neo}$. This would be the generalisation of Eq. (9.25) to plasmas with temperature gradients.

XII. Summary and conclusions

The bootstrap current in toroidal systems is the result of the tangential force balance within the magnetic surface. The pressure tensor, which by various mechanisms - either particle drift in the inhomogeneous magnetic field or by external heating mechanisms - has become anisotropic, leads to tangential forces on the magnetic surface. In steady state these forces have to be balanced by the friction forces of relative mass flow within the magnetic surface. The coupling to the radial diffusion flux originates from the Lorentz force $\mathbf{v}_1 \times \mathbf{B}$ ($\mathbf{v}_1$ is the local radial diffusion velocity) which has to be balanced by the perpendicular friction force between different particle species and the perpendicular component of $\nabla \cdot \pi_j$. The flux - friction relations are the surface averaged force balance equations which couple the toroidal and poloidal fluxes to the radial diffusion fluxes. These equations are linear algebraic equations between these fluxes and can be used to express the toroidal bootstrap current as a linear function of the radial particle flux and the electron thermal flux. In toroidal systems with one ignorable coordinate - tokamaks and quasi-helical stellarators - a simple equation between bootstrap current and neoclassical fluxes exists. This is the generalisation of the relation which was found by Bickerton et al. Ref.[9] . In these symmetric configurations the bootstrap current is always finite; the factor which relates the current to the radial fluxes does not depend on the details of the magnetic surface. In non-axisymmetric configurations this is different since, depending on the specific Fourier spectrum of $mod\,B$ on every magnetic surface given in Hamada coordinates, the bootstrap current can have either sign and can even be zero on nearly all magnetic surfaces. In a torsatron, where the helical harmonics increase towards the edge, zero bootstrap current can occur on one specific magnetic surface. An example of zero bootstrap current on nearly all magnetic surfaces (evaluated in the plateau regime) is given in a 4-period Helias configuration (see Fig. 2).

In the long mean free path regime the coefficients L_{ik} depend on $mod\,B$ and the collisionality and therefore zero bootstrap current is only possible for a specific collisionality and not in a finite region. Furthermore, it has to be expected that the electric field, which enters the kinetic equation via the drift velocity $\mathbf{v}_D$ also determines the bootstrap current. In the very long mean free path regime the coefficients L_{ik} are proportional to the collision frequency and a geometrical factor can be separated which depends on $mod\,B$ and the radial alectric field. If the dependence on the radial electric field is weak, a geometrical factor G_b is a relevant figure of merit for the bootstrap current in the long mean free path regime.

The limits of the theory described above are twofold. First, to neglect the inertial forces in the tangential force balance is only justified if the lowest order plasma flow velocity $\mathbf{V}_j$ is small enough. This can be the case in a plasma without external momentum sources where $\mathbf{V}_j$ consists of the diamagnetic drift and the $\mathbf{E} \times \mathbf{B}$-drift of similar size. The parallel velocity is also of the same order as the diamagnetic velocity. With external momentum sources, however, a strong poloidal rotation may occur and the tangential force balance will be modified by inertial forces. This problem needs further study.

Second, shear and gyro-viscous terms have been neglected in the pressure tensor. Usually these terms are very small, however in symmetric systems, where the bulk viscosity has no force in the invariant direction, these terms introduce a damping mechanism which removes the degeneracy of the flux friction relations. In a collision dominated plasma the full Braginskii pressure tensor can be retained to calculate $< \mathbf{B} \cdot \nabla \cdot \pi_e >$ and $< \mathbf{V}_o \cdot \nabla \cdot \pi_e >$. Shear viscosity and gyro-viscosity provide a coupling between different magnetic surfaces and the resulting flux - friction relations are a second order system of differential equations for $E_j(\psi)$ and $\Lambda_j(\psi)$ instead of the algebraic system described above[1]. In a neoclassical plasma one would expect a strong shear viscosity arising from the finite banana orbits. This effect can be computed by retaining the ψ-derivative in the kinetic operator L. The solution of the kinetic equation then contains radial derivatives of $E_j(\psi)$ and $\Lambda_j(\psi)$ and the surface average viscous forces are linear in $E_j(\psi)$,$\Lambda_j(\psi)$ and their first order and second order derivatives. A calculation in this direction has been made by D.E. Hastings[2] a generalisation to the flux-friction relations is still needed.

A further limit of the theory arises from singularities around rational magnetic surfaces. The flux-friction relations are the necessary conditions for the first order magnetic differential equations for pressure p_1 and electric potential Φ_1 having single-valued solutions. On closed magnetic field lines a further condition exists which generally cannot be satisfied in non-axisymmetric systems and therefore the ordering scheme described in chapter II is not applicable. In the close neighbourhood of rational surfaces viscous forces have to be taken into account in lowest order to remove these singularities.

These remarks indicate the limits of the theory of bootstrap currents. The theory is still incomplete. The experimental test of this theory is overshadowed by anomalous transport phenomena in tokamaks and stellarators, which cast some doubts on the applicability of neoclassical theory to real plasmas. This calls for a modification of bootstrap theory including turbulence effects. A first paper on this subject has recently been published by K.C. Shaing[1].

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